ALLOCATING RESOURCES TO PEOPLE WITH PREFERENCES

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ALLOCATING RESOURCES TO PEOPLE WITH PREFERENCES

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SUMMARY

Resource allocation can be viewed as the assignment of available resources to different populations, projects, or tasks. Usually, resources can be allocated by markets (where people exchange goods or services based on price and personal preferences) or by central planning (a central planner or agency makes the decision). In many industrial applications such as assembling automobiles from parts or assigning time-constrained tasks to processors, central planning can often yield the optimal result with respect to the entire system. However, in other settings such as allocating healthcare services to patients, distributing vaccines to local providers, or assigning jobs to workers, preferences of the individuals need to be considered in addition to system objectives.

Three practical and theoretical problems are presented that involve allocating resources to populations with different preferences or demands: 1) measuring the spatial access of patients across a large network where patients have different preferences over the healthcare providers, 2) quantifying the benefit of allocating vaccines to local areas based on inventory information when vaccine uptake rates vary geographically, and 3) assigning workers to jobs in a multi-period setting with workers and jobs listing preferences over each other. We use mathematical modeling approaches tailored for each of the problems that allow us to capture the preferences of populations in each system.

Chapter I provides a brief introduction to the three research problems. Chapter II presents an optimization framework for measuring healthcare spatial access, where patients' choice over service providers are incorporated using notions of equilibria. We analytically demonstrate the advantages of using optimization approaches to quantify patients' access to service providers and illustrate these advantages via a case study using data on Cystic Fibrosis in the United States. Chapter III addresses the value of inventory information in distributing vaccines in a flu pandemic, where using vaccine allocation strategies that utilize inventory information can improve the percentage of demand satisfied, reduce

the disease incidence, and decrease inventory of wasted vaccine. Chapter IV introduces modeling for stable matching between workers and jobs in a multi-period setting, where we discuss new notions of stability and model the problem using integer programming with the preferences of people either staying the same or changing over time. We obtain theoretical bounds on the objective function value under special preference lists and develop fast heuristics to solve the problem with large number of agents and small number of periods. In addition, we provide insights on the sufficient and necessary conditions under which our algorithms and heuristics work well. In conclusion, the problems presented in the thesis demonstrate how mathematical modeling can be used to incorporate different types of preferences in a network and to inform efficient decision makings for complex systems.

CHAPTER 1

INTRODUCTION AND BACKGROUND

The problem of effectively allocating resources to populations with different preferences arises in many practical situations. In some applications, a central planner allocates resources based on available information and tries to optimize the allocation for the entire system. However, in other settings such as allocating healthcare services to patients, distributing vaccines to local providers, or assigning jobs to workers, preferences of the individuals have to be considered besides the general welfare of the entire system. We address three practical problems involving allocating resources to people under preferences of different kind.

Patients often have preferences over the location of healthcare providers based on distances, waiting time, or other factors. Government agencies such as the U.S. Department of Health maybe interested in assessing the status quo of healthcare access to identify populations living in under-served areas. Potential interventions could be implemented in the future to improve populations' access to care in these areas.

Usually, the current choices of patients (utilization of services) maybe influenced by the locations of existing facilities (e.g., patients may choose to move for better access of care), which may not reflect the underlying access to care. Instead, potential access is often used in the literature to infer patients' access to care in current and future health networks. Existing methods for quantifying access include using simple physician-to-patient ratios or catchment methods incorporating congestions, but neither of the existing approach considers adjusting for changes in the entire network or population preferences.

In Chapter II, we present decentralized optimization as a framework to measure patients' access to care, incorporating system effects and population preferences. The optimization methods assign patients to provider locations, minimizing the total distance traveled and congestion (or waiting time) experienced while satisfying different patients' demand. We use a Nash equilibrium condition for each visit of a patient to ensure that the patients will not deviate from their current choices of providers locations. We analytically demonstrate the accuracy of decentralized optimization compared to the two-step floating catchment area method and its variations. We also use a case study of specialty care for Cystic Fibrosis over the continental United States to compare these approaches.

Our analysis shows that the catchment methods overestimate patients' demand, especially in dense urban centers and areas with overlapping health service providers. The catchment methods also capture patients' opportunity instead of experience and have no mechanism to capture cascading effects based on congestion. Furthermore, the optimization models account for more elements of access (e.g., capacity of each provider and patient preferences) than the traditional catchment methods. Finally, optimization models can incorporate user choice and other variations, and they can be useful towards targeting interventions to improve access. The study is published in BMC Health Service Research [1].

In Chapter III, we study the value of inventory information in allocation of flu vaccine with limited supply. Influenza in the US can result in thousands of deaths annually, while timely vaccination can prevent the disease. However, flu vaccine supply is often limited because of the production process, especially lead time. At the same time, people's willingness to receive a vaccine varies geographically, which is quantified as uptake rate at local levels. During the 2009 H1N1 pandemic in the United States, a federal agency (the Centers for Disease Control and Prevention (CDC)) coordinated the distribution of limited vaccine and allocated vaccine proportional to population of states when available (pro-rata). At the beginning of the pandemic, there was limited vaccine available. However at the end of February 2010, only half of the available vaccine was administered, resulting in millions of vaccine being discarded. Reporting of vaccine administration was required during the 2009 H1N1 pandemic, although only a few states chose to track detailed information, such as the geographical location. With the addition of this information to the vaccine registry, it would be possible to allocate vaccines not only by population, but also according to demands from local areas.

We use a detailed agent-based stochastic model to simulate the spread of the disease and study the allocation of limited vaccine supply when the uptake rates vary geographically. We allocate vaccine proportionally to locations when there is no inventory information (*Population-Based*, or PB strategy). When there is inventory information, we allocate vaccine proportionally to locations that are continuing to take vaccine (*Population-and-Inventory-Based*, or PIB strategy). Using the state of Georgia in the United States as a case study, our research shows that using strategies that incorporate inventory information can reduce the number of flu cases, decrease wasted vaccines, and satisfy higher percentage of demand. Our results emphasize the need for greater visibility in public health supply chains, which could be achieved through additions to the existing vaccine registry information systems.

Finally in Chapter 4, we study a multi-period matching problem with two-sided markets, complete preference lists, no ties, and relaxed stability. Organizations often need assignments that can reflect the preferences of workers and jobs (stability), while at the same time promoting utilities among all workers (quality) to reduce turn-over rate. More often than not, decisions have to be made across multiple periods to allow organizations more flexibility in making assignments and negotiating with workers. Our research goal is to help organizations operationalize their decisions based on the desired level of stability and quality of assignments.

The notion of stability comes from the stable matching model studied by Gale and Shapley [2]. In a single period, a matching is stable when any worker-job pair does not have unilateral incentive to deviate from the current choice. To measure the quality of matching, a utility function is often applied for each worker-job pair depending on the positions of jobs on workers' preference lists. The stable matching concept is widely used because of its elegant theory and applicability, although it does not necessarily provide matches of good quality for all workers.

We study matchings in a two-sided market in a context of assigning workers to jobs over multiple periods. In our research, we analyze the trade-offs between stability and quality of assignments. We focus on the case when the number of workers and jobs is large (hundreds or thousands) and the number of periods is small (three or four years) as the rationality of individuals are often limited. The objective function is to maximize the minimum individual utility summed over all periods as a measure of quality of the matchings (similar to the min-max regret measure in [3, 4, 5, 6]). We introduce a definition of multi-period stability by aggregating stability constraints in each period and relaxing the stability requirement. Furthermore, we develop heuristics to help organizations operationalize decisions and provide insights on the trade-offs between stability and the quality of matchings. In addition, we explore settings where workers' preference lists change over time and provide modified formulation and heuristics. At last, we characterize the set of preference lists for which our heuristics return a stable assignment.

Overall, our research provide heuristics that solve large scale problems fast, and we demonstrate that the performance of these heuristics is good in terms of improving the quality of matchings from workers' perspective. We also obtain insights from the structure of solutions, which could be useful for organizations to implement suitable decision making tools under various preference settings.

CHAPTER 2
AN OPTIMIZATION FRAMEWORK FOR MEASURING SPATIAL ACCESS OVER HEALTHCARE NETWORKS

2.1 Background

Access to healthcare is widely recognized as essential for ensuring not only care of immediate health needs but also to enable health and wellness in the population. Access has multiple dimensions including accessibility, availability, affordability, accommodation, and acceptability [7, 8, 9] and is of great importance to decision makers in public health. We focus on measurement models for spatial access over a health network with patients and providers, which is most closely related to the elements of accessibility (e.g., location and travel distance for care) and availability (e.g., coverage or the volume of providers). A healthcare network is defined as a transportation network with patients as demand nodes and providers as supply nodes, and an arc between patient and provider if the provider is accessible for the patient.

The measurement models we studied are designed to measure potential access based on the services that are available for use relative to population and distance. On the contrary, realized access reflects actual use of services, which can be affected by finances, behaviors, and other factors. Potential access is measurable although it is not observable. An optimization-based approach is described for quantifying potential access over the healthcare network and for estimating the impact of changes to the network. Optimization is a mathematical science that is widely accepted in engineering and science as providing a way to balance complex interactions across a system, and there is a history of using optimization to assist medical decision making [10, 11, 12]. Theoretical and practical optimization modeling techniques are used to assist with health care policy development by measuring access and computing the economics behind discrepancy of access. Specifically, questions such as how optimization models can be used to measure access, on what types of networks they offer the most accurate estimates of access, and ultimately, why they should be used for measuring and for suggesting interventions to improve access are addressed. The answers to these questions are useful for improving the health of populations and assisting with health policy development by informing areas of greatest need.

The optimization models are compared to some existing methods. In particular, comparisons are made to variations of the two-step floating catchment area (2SFCA) method [13], including the Enhanced 2SFCA (E2SFCA) method [14] and the Modified 2SFCA (MSFCA) method [15], with some discussion of other catchment methods. The catchment methods, which are offsprings of a gravity model of attractions between populations and providers, estimate the size of population served at each provider using distance zones and compute accessibility of a community based on the availability of providers in the communitys zones; communities can be captured in the zones of multiple providers. In contrast, optimization models match patients and providers based on both distances and the relatively crowdedness of each provider, and estimate the accessibility of a patient using the matching results to determine the travel distance and the corresponding crowdedness of each patient. Optimization models can take on the perspective of a centralized planner in making assignments, or they can be adapted to directly incorporate patient choice over the network. To compare the measurement models for spatial access, several specific network structures are examined, which are designed so that access measures can be compared analytically. Results on a large case study of specialty care of Cystic Fibrosis (CF), where the network has varying levels of accessibility are also provided.

Analytically, we demonstrate that the total number of patient visits captured by all facilities in the 2SFCA methods is larger than the number of visits expected based on population size. The three-step floating catchment area (3SFCA) method [16] adds an assignment mechanism to address the competition by facilities, but the assignments are only based on distance. In contrast, in the optimization models, the willingness to travel is not only a function of distance but also of facility congestion including its size. As a result, the optimization models can capture cascading effects in the system, where a change in congestion for one population leads to different decisions and thus impacts individuals in another location. The optimization models also allow for simultaneous estimation of measures of access across the five dimensions outlined [7].

More generally, optimization models can be adapted to many contexts including different patient types (e.g., Medicaid or not), provider constraints, or others. They are also useful in optimizing interventions, where the intervention can target different aspects of access (e.g., distance versus congestion).

2.2 Methods

2.2.1 Optimization Framework

In healthcare decision making and service research areas, optimization models have been used to determine the best location for a new clinic [17, 18, 19], ensure that resource locations are sufficient to cover the need across a network [20], route nurses for home health services [21], improve health outcomes among communities [22, 23], and evaluate policies for pandemic influenza, breast cancer, and HIV over a network [24, 25, 26], among others. Wang [27] reviewed several cases where optimization models could be used to improve access or service over a network.

In our models, the cost of an individual is associated with two dimensions of access [7]: accessibility and availability. The first is measured with travel distance (or time). The second is measured with congestion, which for an individual is associated with the relative number of people (or visits) at a provider compared to the resources available. One can also think of this as capturing the waiting time until an appointment is available. Studies show that individuals are willing to drive further to receive an appointment more quickly [28]. Thus we assume that the utility (or disutility) associated with a patients access is a weighted sum of the distance and a congestion term, where we scale the congestion term to trade-off the relative importance between the two. We expect that the congestion weight (α) may be different for different types of healthcare services, such as primary care or specialty care (i.e., distance may have a relatively lower cost). The congestion weight can also represent the resources available at a facility.

Several elements are defined for our formulation. The total number of patients is n and the total number of facilities is m. Let $i \in [n]$ be the indices of patients and $j \in [m]$ be the indices of facilities. The distance between patient i and provider j is d_{ij}; v_i is the estimated number of visits that patient i will make (demand); and a_j is the congestion weight at provider j. A dummy location can be introduced for the assignment of demand that cannot be met.

The decision variables are x_{ij}, which is the percentage of time assigned to facility j from patient or community i, for each $i \in [n]$ and $j \in m$. The formulation of the basic centralized model follows:

Formulation 1 Centralized spatial access optimization model

$$\min \quad \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}x_{ij}v_i + \sum_{j=1}^{m} \alpha_j \left(x_{ij}v_i\right)^2 \tag{2.1}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} x_{ij}v_i = v_i \qquad \forall i \in [n] \tag{2.2}$$

$$0 \leq x_{ij} \leq 1 \qquad \forall i \in [n], \forall j \in [m] \tag{2.3}$$

The objective function (2.1) states that the total number of visits assigned should be v_i for each patient or community i. Constraint (2.2) requires that all individuals be assigned, and equation (2.3) requires non-negativity of the decision variables. Each individual's congestion at a visit is proportional to the total number of visits at that facility scaled by α_j. The congestion term in the objective sums over the congestion experienced by all patients resulting in an overall term that is squared. The choice of quadratic function comes from the following idea: if n patients receives care from a provider location, then each patient experiences n units of congestion, then the total congestion is $n \times n = n^2$. Note that when $\alpha = 0$, this model gives equivalent results to assignment by shortest distance, and when $\alpha = \infty$, this model gives equivalent results to equally distributing patient visits to each facility. See section A.1 for a process to select the congestion weight.

For a patient, the number of visits to a close location is expected to be more than the number of visits to a far location because of the willingness to travel. Thus, the number of visits to each location using a function that decays with distance is determined. This is analogous to step 1 in the E2SFCA method where the population is multiplied by a weight. This also implies that the number of visits covered in the network may be less than 100%.

From the results of an optimization model, several measures of spatial access are calculated. The measures include i) the distance traveled for each patient or community; ii) the congestion experienced by each patient or community; iii) the coverage, which is defined as the ratio of visits assigned to visits needed for an individual or community.

2.2.2 Variations on the Optimization Model

With optimization models, many variations are possible, including through the addition of constraints, the use of different objective function values, or by differentiating decision variables by type. Here we describe a major variation in our model, *optimization with user choice (Decentralized)*, and include many others such as capacity, unmet demand, and willingness to travel in section A.2.

The traditional deterministic optimization model (as presented above) often assumes a centralized planner who makes decisions for every patient in a healthcare network to achieve the best overall objective. However, user choice can be incorporated by an equilibrium constraint that represents individual choices as in game theory [29]; we call the resulting optimization model decentralized.

An overall equilibrium solution requires a user choice constraint to be satisfied for each patient visit in the network, where the constraint states that the individual cannot improve their distance and congestion of that visit by switching to another facility given the other decisions on the network. The decision variable and equilibrium constraint are defined below: x_{ijk} = decision variable is 1 if patient i chooses facility j for visit k, or 0 otherwise;

$$d_{ij} + \alpha_j \sum_{p=1}^{n} \sum_{k=1}^{v_p} x_{pjk} \leq d_{iq} + \alpha_q \left(\sum_{p=1}^{n} \sum_{k=1}^{v_p} x_{pqk} + 1 \right), \forall q \neq j, \forall i, \forall k \tag{2.4}$$

The equilibrium condition includes a separate constraint for each patient's visit and each location when there is no distance decay function. The left-hand side is the distance and congestion associated with current facility choice j for a visit k, and the right-hand side is the distance and congestion at any location other than j. See section A.3 for more details.

2.2.3 Review of Catchment Models

Gravity models use the following general form to calculate an attraction measure for each patient i:

$$A_i^G = \sum_{j=1}^{m} \frac{S_j w(d_{ij})}{\sum_{i=1}^{k} p_i w(d_{ij})} \tag{2.5}$$

where S_j is the supply at provider j, P_i is the population at location i, $w(d_i j)$ is the decay function based on distance of each patient-provider pair (i, j).

The original 2SFCA method was introduced by Luo and Wang [13]; it allows the catchment of each provider and patient to float based on the distances between each pair. E2SFCA is a variation that suggests applying different weights within travel time zones to account for decaying of the willingness to travel as distance increases [14]. Under the E2SFCA model, in the first step the physician-to-population ratio at each provider is calculated. Although the E2SFCA aims to estimate the number of patients that may potentially use a facility, it is easy to extend the metrics to estimate the number of visits by replicating each patient using visits demanded (e.g., a patient demanding 10 visits can be viewed as 10 patients) [30, 31]. We make a minor adjustment to allow for each patient to have multiple visits to a provider, so we use physician-to-visits ratio instead. Thus we obtain:

$$R_j = \frac{S_j}{\sum_r \sum_{i \in \{D_{r-1} \leq d_{ij} < D_r\}} V_i W_r}, \tag{2.6}$$

where S_j is the number of physicians available at provider j; W_r is the weight value corresponding to the catchment zone of d_ij. The value of W_r is calculated using the distance decay function, which is usually nonlinear; D_r is the distance threshold of catchment zone r. The parameter V_i is the number of potential visits if there is no decay in willingness to travel or the maximal demand for patient or community i. The original E2SFCA method introduced the model with three catchment zones, but an extension is to allow a different number of zones or even a continuous decay (impedance) function across a single zone. Example choices of impedance functions include Gaussian [13, 32] , exponential, inverse power, and others; [32] discusses parameter setting for the impedance function.

In the second step of E2SFCA, the method defines the accessibility of each patient or community i based on the ratios at each provider and the zone weights:

$$A_i = \sum_r \sum_{j \in \{D_{r-1} \leq d_{ij} < D_r\}} R_j W_r. \tag{2.7}$$

Another catchment approach is the 3SFCA method, which incorporates competitions among multiple providers within the same catchment zone of a patient and makes assignments of patients by distance. The M2SFCA method [15] modifies the patient level accessibility in [13] by multiplying the distance weight twice, while another approach [33] allows for zones to differ by transportation modes.

For a simple system, the individual measures of spatial access from optimization models can be combined to directly compare with the accessibility measures of 2SFCA methods (E2SFCA and M2SFCA). The simplest supply network consists of n communities in a circular population area with a facility at the center. Let d_i be the distance from community i to the facility and S the number of physicians in the facility. Calculate the facility population-to-physician ratio R and patient accessibility A_i using equations 2.6 and 2.7. Define a decay function $w(d_i) \in [0, 1]$. For this system, the optimization method is equivalent to assigning by shortest distance. Let F denote the congestion at the facility, then $F = 1/R$. The coverage of community i is calculated as $w(d_i)$. Therefore, for this system, the patient accessibility is $A_i^E = \frac{coverage}{congestion}$, for the E2SFCA method. For the M2SFCA method, a similar calculation can be made, where the composite patient accessibility measure is $A_i^M = \frac{coverage^2}{congestion}$.

2.3 Results

2.3.1 Analytical Comparisons

In this section, analytical results on accessibility as measured by the optimization method and catchment models are provided. Most analyses in this section focus on simple systems where service areas are non-overlapping. For simple networks with overlapping service areas, the detailed analysis can be found in section A.4. Notations that will be used frequently in the analysis are defined below. The distance decay function $w(d_{ij})$ is between 0 and 1. If d_{ij} is the distance between community i and facility j, and v_i is the visits needed by community i, then we assume that facility j receives $w(d_{ij})v_i$ visits from community i as in the catchment models. In optimization models, let p_{ij} be the proportion of the population in community i that visits facility j.

Result 1 (Opportunities vs. Experiences): Optimization models capture a patient's experience rather than their opportunities. As a result, 2SFCA methods tend to overestimate the total number of visits.

For many catchment models, the estimated accessibility measure increases when more facility choices are available to a population. However, assignments models (including optimization and the 3SFCA method), are estimating the cost of potential access, and this does not increase if a new choice is congested or inconsequential. This is illustrated with a simulated system of populations and facilities, as in Delamater (2013) [15] . Consider

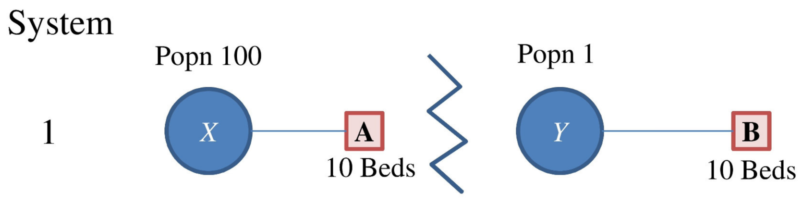

Figure 2.1: System 1, with populations 100 at location X and 1 at Y. Facilities A and B each have 10 beds.

System 1 as described in Figure 1. When facility A and population X are sufficiently far from B and Y, the catchment models and the optimization method will provide the same accessibility estimate. Consider a second system, where B and Y are both closer to X and A than in the first system, with the distances between $A - X$ and $B - Y$ retained and b closer to Y than A. The 2SFCA methods show that the accessibility of Y increases due to the possibility of service at A, while the accessibility of X decreases because of demand on facility A from population Y. However, the optimization method shows there is no change in accessibility for reasonable congestion weights. From the perspective of a person at Y, service at facility A would be associated with a higher congestion cost and a further distance, thus he would neither be assigned to facility A nor choose that facility. This is still the cost associated with potential access rather than realized access, but the cost is associated with the potential experience of a patient. In contrast, the 2SFCA methods always realize additional choices regardless of their relative competitiveness to existing choices. Therefore the total number of visits implied by the 2SFCA methods is higher compared to the optimization method, and can be higher than the total number of visits demanded.

Result 2 (System Effects): The 2SFCA methods do not capture the cascading effects based on congestion.

For methods focused primarily on catchment zones without assignment, there are some system effects that may not be captured over the network. To illustrate this, define System

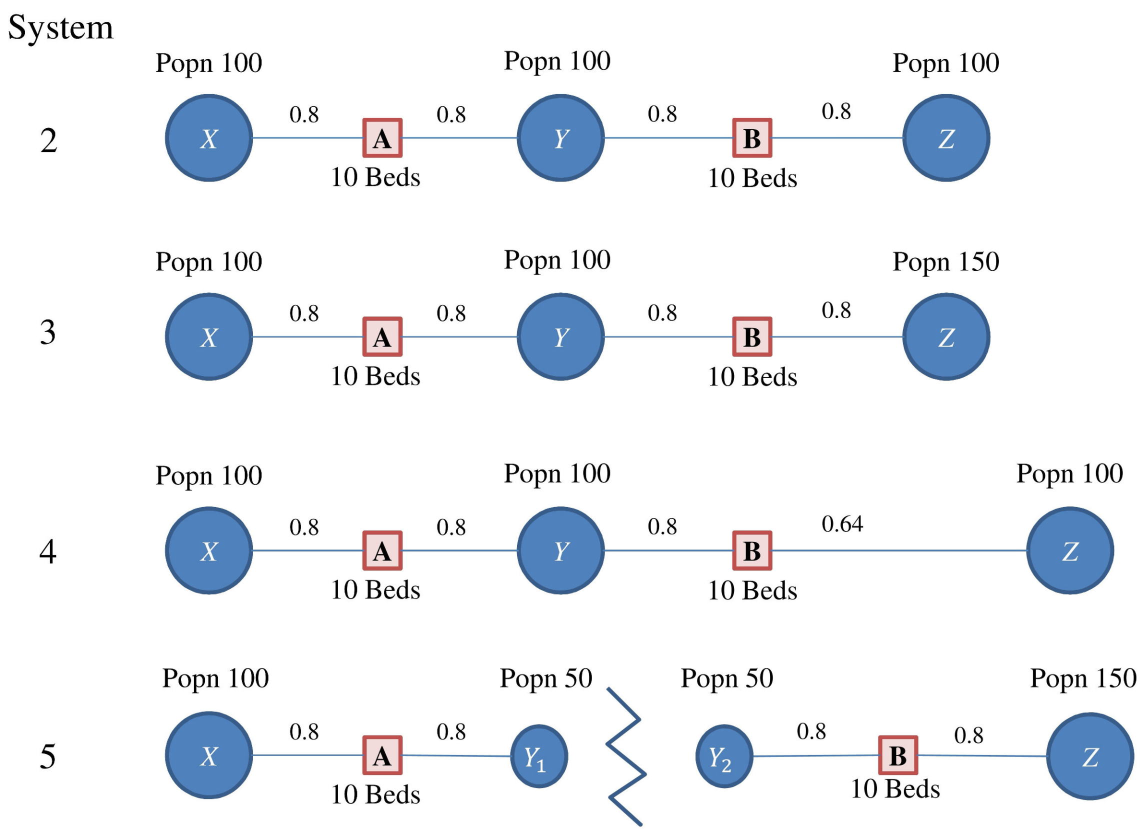

Figure 2.2: Systems 2 through 5, with populations as specified at location X, Y, and Z. Facilities A and B each have 10 beds, and the distance weights are provided between locations.

2, with population z added to system 1, and with a population of 100 for each of X, Y, and Z. In this system, the optimization method and the 3SFCA both compute the same accessibility for each population, while in the 2SFCA methods the accessibility is higher for Y since it is capturing opportunities for access rather than the patient experience.

Consider System 3 with increased population at location Z. In the catchment models, as the population of Z increases, the accessibility for Y and Z decrease, while the accessibility for X remains the same no matter how large Z is. In the optimization method, as Z gets larger, more of the population from Y goes to facility A, so the accessibility at all population locations decreases. The accessibility at each location is the same because the system is constructed in a very specific and symmetric way.

A similar effect can be seen when System 2 is varied by moving population Z further away from the center (System 4). In this case, more patients from Y switch to B to reduce congestion, resulting in better access for population X in the optimization method, while the 2SFCA methods show no change for X.

Define System 5 the same as 1 but with an unbreakable barrier separating population Y in half, and a population of Z equal to 150. The 3SFCA quantifies the same access with and without the barrier, because the assignment is based on distance alone. On the other hand, the optimization method shows different access in System 5 compared to 3, because assignment is based on both distance and congestion. The accessibility estimates for the different systems are summarized in Table 2.1. **Result 3 (Composite Measures vs.**

Table 2.1: Accessibility estimates for systems 2 to 5.

	E2SFCA			M2SFCA		
System	X	Y	Z	X	Y	Z
2	0.05	0.1	0.05	0.04	0.08	0.04
3	0.05	0.083	0.033	0.04	0.067	0.027
4	0.05	0.106	0.044	0.04	0.084	0.028
5	0.067	$Y_1 = 0.067, Y_2 = 0.05$	0.05	0.053	$Y_1 = 0.053, Y_2 = 0.04$	0.04
	Optimization A^E			Optimization A^M		
2	0.067	0.067	0.067	0.053	0.053	0.053
3	0.057	0.057	0.057	0.046	0.046	0.046
4	0.071	0.071	0.071	0.057	0.057	0.037
5	0.067	$Y_1 = 0.067, Y_2 = 0.05$	0.05	0.053	$Y_1 = 0.053, Y_2 = 0.04$	0.04

Individual Measures): The composite measures of the 2SFCA methods are insufficient to distinguish multiple elements of access. Consider systems 6 to 8. System 6 has 100 people in X and 10 beds in A, and the distance weight between X and A is 0.1. System 7 is similar to system 6 but with a distance weight 0.2 (which implies the population is closer to the facility). System 8 is similar to system 7 but has 5 beds in A. As we move from system 6

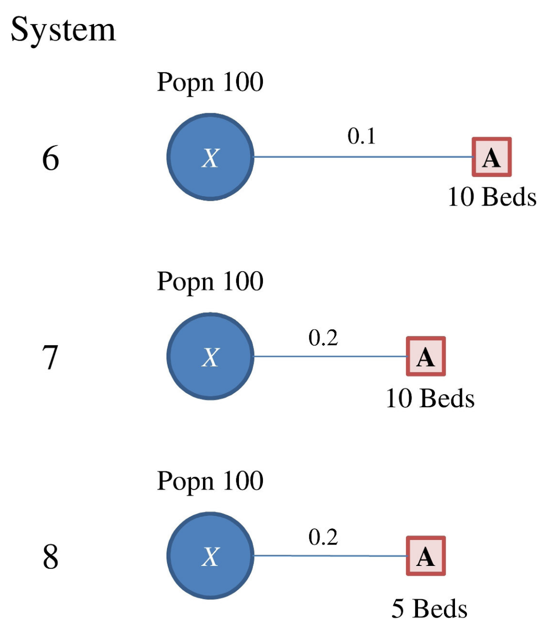

Figure 2.3: Systems 6 to 8, with population of 100 at location X, and a single facility with either 5 or 10 beds. Distance weights are provided for each system.

Table 2.2: Accessibility estimates for systems 6 to 8.

System	E2SFCA	M2SFCA	Opt Coverage	Opt Congestion
6	0.1	0.01	0.1	1
7	0.1	0.04	0.2	2
8	0.05	0.01	0.2	4

to system 7 and then to system 8, either the population is closer to the facility, the facility has fewer beds, or both, so the network is getting more congested and the accessibility of X should reflect this change. However, as Delamater [15] points out, the E2SFCA method shows the same accessibility for populations in system 6 and 7. Similarly, the M2SFCA method shows the same accessibility for populations in system 6 and 8.

The individual measures in the optimization method indicate the coverage increases as you move to system 8 but that the congestion also increases (see Table 2.2).

2.3.2 Case Study

The analytical analysis above illustrates several direct comparisons between the 2SFCA methods and the optimization method. In this section access is estimated for the specific health service network associated with Cystic Fibrosis (CF), which is a chronic condition that requires specialty care. Recent studies have shown that Medicaid status is related to survival rate and outcomes [34], but spatial access may also be a factor. The condition has prevalence in the United States of about 30,000 patients with 208 CF care centers in the continental US [35]. Though it is a rare disease, the service network displays heterogeneity, with the spatial access varying greatly over the network.

Focusing on potential spatial access, locations of CF patients are simulated according to the incidence of the disease rather than using existing locations of actual patients (which may be biased by service locations). With CF, the population eligible for Medicaid is considered separately, since they may need to receive service in their home state. 30,000 virtual patients are generated with CF located in county centroids in the continental US, where the prevalence was generated proportionally to the populations in each race/ethnicity who are above or below 2 times the federal poverty level [36], using the incidence matrix for race/ethnicity in section A.5 Table A.1. Patient demand is defined as 10 visits per year to a center (which captures more than 90% of the patients with location information available in the CF Foundation Registry data [35]). We assume the actual number of visits is decreasing with the distance to selected service facility, patients will not visit facilities more than 150 miles away (again, this captures more than 90% of the patients in the registry with location information [35]). We also assume that low-income patients will only visit a CF center within the patient's state due to restrictions of the Medicaid program.

The zip code of each CF center (see section A.6) is obtained using patient encounter data from the CF Foundation [35], and the road distance from each CF virtual patient to each CF center is computed using Radical Tools [37]. We assume all facilities are the same size (e.g., can serve 1500 visits a year); the exact number can be changed and the relative comparisons between methods will hold.

Accessibility measures were calculated for E2FSCA, M2SFCA, and the decentralized (with user choice) optimization model. The optimization model was implemented using C++ and the CPLEX solver on a UNIX system. The decay functions are such that 10 visits will be made when distance is zero, and visits approach zero when distance is 150 miles; see specific functions in section A.7 Table A.2. There are many functions that can be used to model the decaying willingness of travel. We have chosen to use the exponential function for the rare disease setting of Cystic Fibrosis. Because CF is rare and access to care is relatively low compared to primary care, patients are willing to travel longer distances than for some conditions. The parameter used in the case study was calibrated to be in line with the CF registry data (see section A.6 Figure A.5. For the optimization model, a congestion weight of 10 is used unless otherwise specified (see section A.1). For the 2SFCA methods, Medicaid patients were only included in catchment areas of facilities in their own states.

Maps of the decentralized optimization model display the distance traveled and the congestion experienced by each person, averaged at the county level, in Figure 2.4(A) and 2.4(B) (uncovered demand is shaded in both maps, and centers are marked with pins). In general, distance is small close to centers, especially in areas with multiple centers such as the coastal northeast. There are a few pockets with higher distance, especially in parts of the West. Congestion is higher in a few areas, such as around Houston and some parts of Ohio and Pennsylvania. Some counties have no simulated patients, while others have uncovered demand, such as in many counties in the Midwest or Western regions. There are also isolated areas that are uncovered, such as near southwest Georgia, southern Missouri, and some counties at the boundary of the US. A summary histogram is provided for distance, congestion and coverage for each county in section A.6 Figure A.4. The distribution of coverage shows that many needed visits are not met, due to the distance patients need to travel to CF centers.

The composite measure A^E generated from the decentralized optimization model is

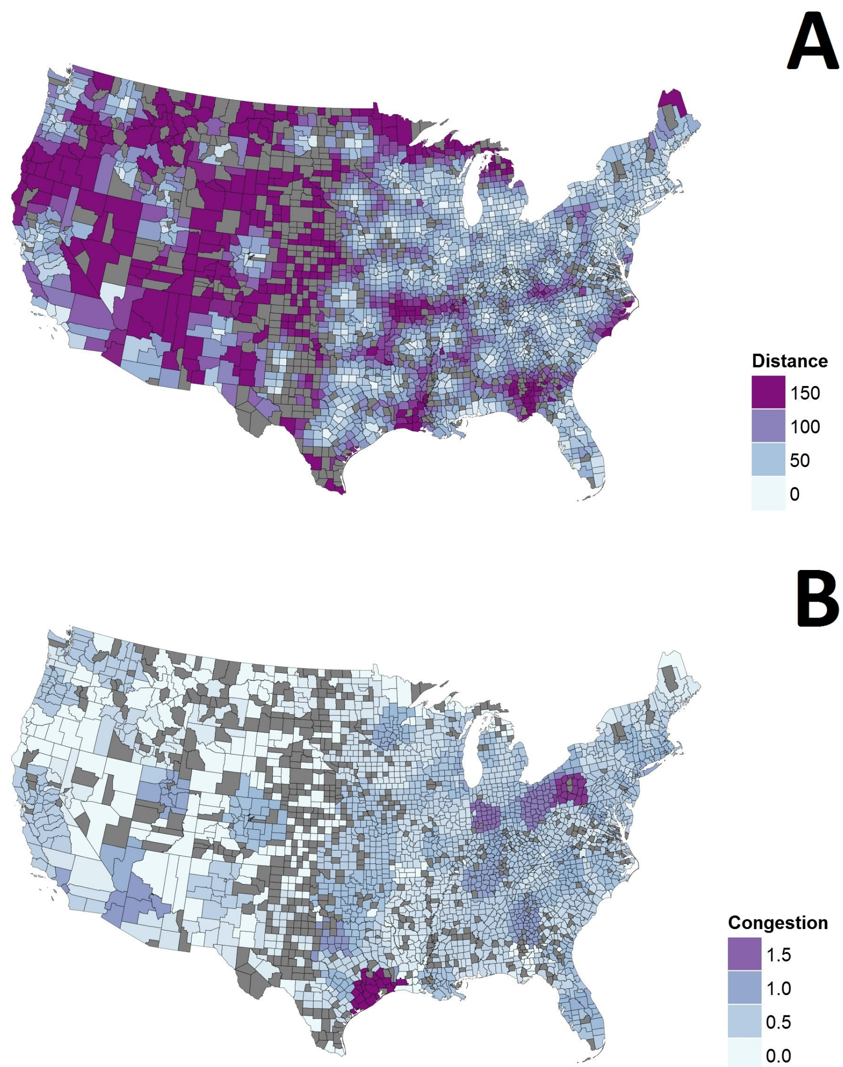

Figure 2.4: Optimization results for patient cost of potential access. (A) Distance, and (B) Congestion.

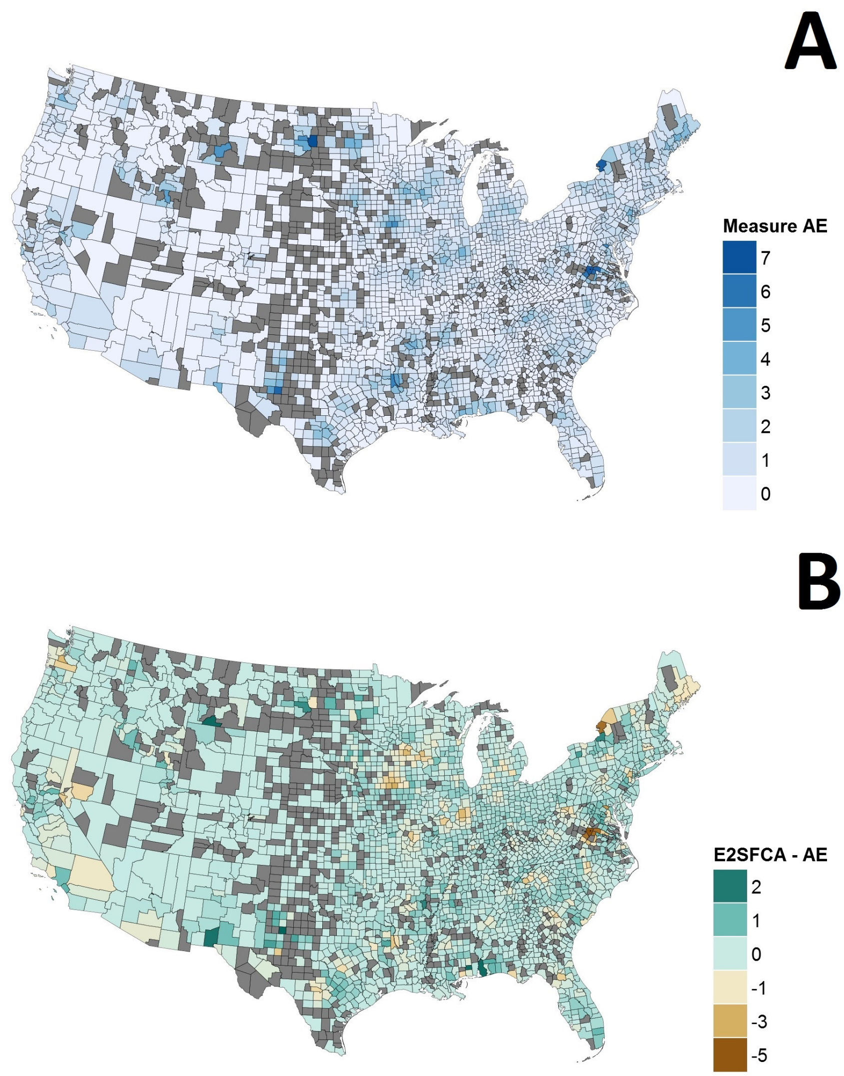

Figure 2.5: Results comparing optimization model with E2SFCA and M2SFCA for CF care in US. (A) Decentralized model composite measure A^E, , (B) E2SFCA-A^E.

shown in Figures 2.5(A). The main areas with high accessibility are near CF centers and around urban areas. There are pockets of low accessibility in many places; however, these can occur for different reasons. In Pittsburgh, Pennsylvania, and Columbus, Ohio, Figure 2.5(A) shows that the congestion was high, while in Springfield, Missouri, Figure 2.5(A) shows that the travel distance is high. Pockets of low accessibility in New York arise from a combination of longer distances and higher congestion.

Figure 2.5(B) show the difference between the decentralized optimization model composite measures and the E2SFCA method using the same scale. In comparison to the optimization approach, the E2SFCA method tends to show higher accessibility in areas with many centers (e.g., near Los Angeles and around New York). It also shows higher accessibility in many areas that lie in overlapping service areas for centers (e.g., northern South Carolina, eastern Arkansas, and New Mexico). A pairwise t-test (1-tail) shows that for counties with more than 50 patients (127 large counties) or less than 5 patients (1289 small counties), the measure from the E2SFCA method is significantly higher than measures from the optimization method (respectively, with p-values 0.20×10^{-6} and 2.00×10^{-2}); for counties of other sizes (medium counties), the test is inconclusive. The F-test shows that for all groups of counties, the variance of the E2SFCA measure is higher (with $p = 1.88 \times 10^{-4}$ for small counties, $p < 10^{-6}$ for medium counties, and $p = 3.90 \times 10^{-2}$ for large counties). The MannWhitney-Wilcoxon test shows that E2SFCA measure is greater in location shift than the optimization composite measure with $p < 10^{-6}$ for small and medium counties, and $p = 2.02 \times 10^{-2}$ for large counties. The finding is consistent with the analytical results in section A.4 showing that with overlapping catchment areas, E2SFCA quantifies higher access when distances are relatively small. The comparison between A^M and the access measure of the M2SFCA method is similar but with a smaller difference.

The number of visits captured in the E2SFCA methods is shown in Figure 2.6 in comparison to the visits needed by the population. It is highest around facilities, and especially with multiple facilities such as around New York. For the optimization model, the realized

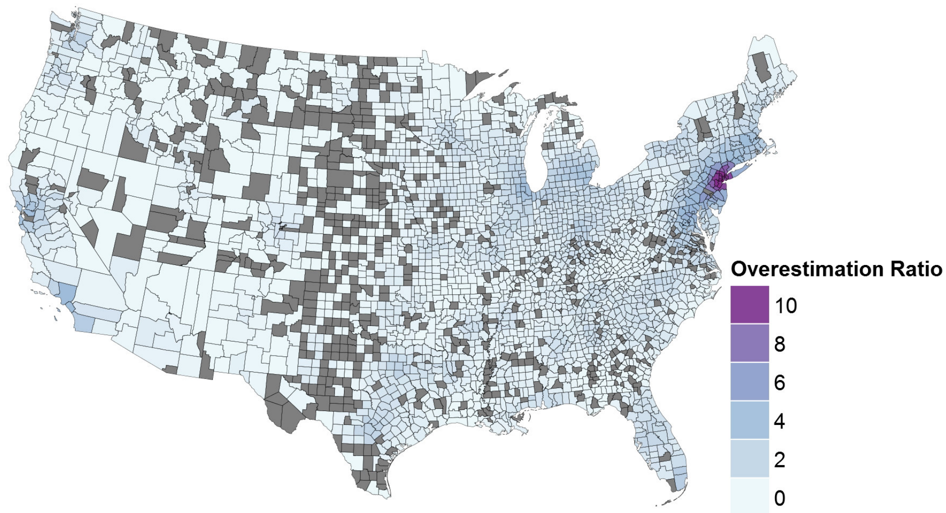

Figure 2.6: Estimated patient visits in E2SFCA and M2SFCA relative to the visits needed in each county. A value greater than 1 indicates that the 2SFCA methods estimate more visits than needed.

visits per facility are estimated to be 0 to 3,000. In contrast, the range for the E2SFCA result is 0 to 10,540 per facility. The F-test indicates that the variance of the facility congestion is significantly higher for the E2SFCA approach, with a p-value less than 10^{-6}. This is similar to the analytical result that the optimization model always has a lower facility congestion.

The results showing access over the network indicate a number of areas that have uncovered populations, high congestion, and/or high travel distances. Figure 2.7 shows the results in several local areas after network interventions. One new facility was added to the network in locations with uncovered populations (Springfield, MO), and the capacity of existing facilities was doubled in two locations (Columbus, OH; and Pittsburgh, PA). For the E2SFCA methods, the gain in access is centered over the interventions and decays with distance within 150 miles. The gain is positive in all areas with change, as the new facilities increase the opportunities available. Under the optimization method, the coverage in an area increases when a new facility is added, and congestion in an area decreases when new capacity is added. Although the total access increases, some populations show a

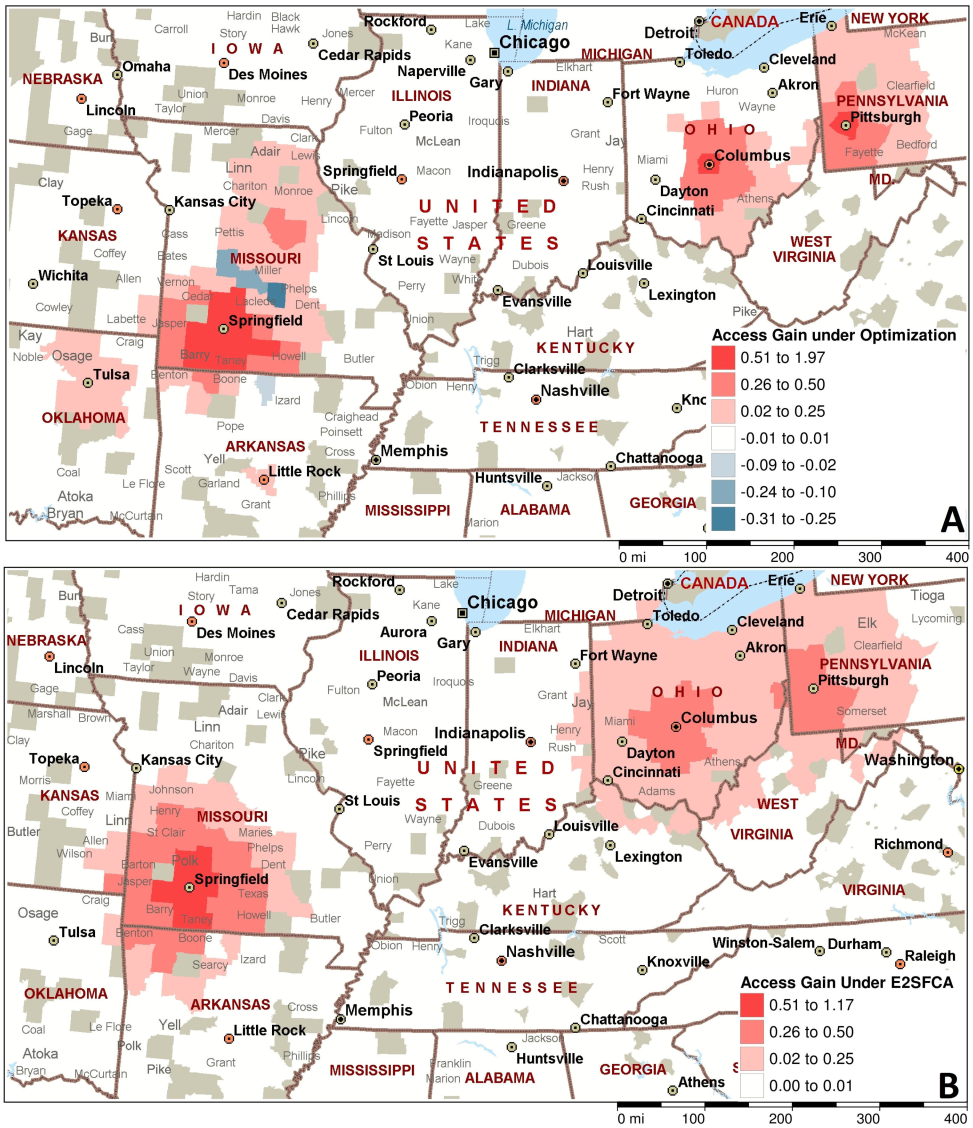

Figure 2.7: Optimization results showing impact of intervention near locations Springfield, MO, Columbus, OH, and Pittsburgh, PA. (A) Access gain under optimization using composite measure A^E, and (B) access gain under E2SFCA.

worse composite measure, which indicates that they are traveling shorter distances but experiencing higher congestion (or the reverse) based on new network dynamics. Note also that when the new location is added in Springfield, there are cascading effects under the optimization approach, and access increases for the population around Jefferson City, since their congestion is decreasing due to the new facility. We performed a t-test comparing the impact of intervention on both measures, and the test shows that the E2SFCA measure changes significantly more compared to the optimization measure, which is consistent with our discussion above.

2.4 Discussion

The optimization method provides several innovations useful both for understanding access and designing interventions. They can be applied across heterogeneous networks with both dense and sparse areas, and they allow user choice to balance travel and congestion within communities. The approach presented includes a way to select the specific parameters of a model. Optimization models also provide both a picture of the status quo and an approach for evaluating a potential change to a network. Fundamentally, the optimization models have a different framework than many catchment methods, since they estimate the access costs associated with a patient's experience (albeit the potential experience rather than actual utilization).

Under optimization models, the presence of additional opportunities only provides gains in potential access when they provider better access compared to existing opportunities, while in the 2SFCA methods, additional opportunities always provides gains in potential access. This difference shows that many 2SFCA methods over count visits when there are facilities with overlapping catchment zones. This effect is stronger in areas with the greatest infrastructure of health services, so interpreting accessibility over a network with sparse and dense areas may not be reasonable. One could adapt the approach by dividing the population by the number of facilities in the zone, or use other adaptations as in the assignment mechanism of the 3SFCA approach. However, these adaptations do not address other issues, such as the cascading effects across the system. The catchment methods tend to capture effects in a defined area, but they do not capture the interactions between areas (or the cascading effects over a network if there are changes introduced) as well as assignment models do. This also means catchment methods may mis-estimate availability across a network for complicated networks.

Using optimization models for estimating access has many other advantages, as they can be easily adapted to measure access for different types of patients, over different provider types, or with capacity constraints in the network. Moreover, decentralized optimization models allow differences in access in rural and urban areas, which arises directly from the trade-off between distance and congestion rather than solely from different distance functions. It is also easy to modify optimization models to determine the best locations for facilities given an existing demand and supply network [18, 19]. The individual measures from optimization models show not only where to intervene, but also points to what kind of intervention is needed (e.g., new location to reduce distance versus more capacity to reduce congestion). This is especially true as one moves beyond just one measure of access like spatial accessibility to consider the other dimensions of access [7].

This study focus on estimating potential health access using optimization models. There are limitations with the approach. The optimization models assume that patients are trading off travel distance and congestion rationally across a network, while in reality there might be many other factors considered by patients. In addition, the optimization models are built using deterministic known data. In the case study, the possibility of using satellite clinics or services provided through telemedicine are not considered. Results are also dependent on the specific decay function chosen. Furthermore, the case study also assumes that the transportation modes used by all patients are the same.

Optimization models come at a cost. They are less familiar to many working in public health or public policy. They can be complex to model or compute, although this may be a matter more of the appropriate training than extensive computing power.

It may be most important to use optimization models when a network has facilities with overlapping zones, when one wants to capture the nuances of access across populations, or when one needs to develop interventions to improve access. We hope that the use of optimization models will provoke more discussion in how to measure access, and ultimately how to improve access, especially in light of the increase in computing power and big data that will be coming online in the US health system.

CHAPTER 3

VALUE OF INVENTORY INFORMATION IN ALLOCATING A LIMITED SUPPLY OF INFLUENZA VACCINE DURING A PANDEMIC

3.1 Background

Influenza in the United States has led to thousands of deaths annually over the past decade [38], and historically, there has been a world-wide pandemic about every 30 to 40 years [39]. The last declared pandemic was due to the H1N1 virus in 2009, leading to an estimated average of 363,550 deaths worldwide [40]. Timely vaccination can prevent the spread and reduce the burden of the disease.

Influenza vaccine supply is often limited, especially during a pandemic [38]; hence, vaccine allocation decisions can play a significant role in the overall impact of vaccination on reducing the disease burden. In the United States, the most recent emergency vaccine distribution campaign (2009-2010) was coordinated by the Centers for Disease Control and Prevention (CDC). As new batches of vaccine became available, they were allocated to each state proportional to the states population (i.e., population based or pro-rata allocation). [41]). The states then distributed the vaccine inventory locally, and providers administered vaccines to individuals.

During the 2009-2010 influenza pandemic, states were encouraged to collect and report data about the vaccines administered to the general population [42]. However, very few states collected detailed information on how many vaccines were administered in each location (e.g., a county or a census tract) [43]. As a result, states did not have a good visibility into the uptake rates and vaccine inventory levels geographically. This lack of visibility in the vaccine supply chain caused some areas to end the influenza season with excess inventory (i.e., leftover vaccine) while other areas (especially those with high uptake rates) experienced unmet demand.

This study aims to quantify the value of information on (or visibility into) vaccine inventory, i.e., tracking inventory levels geographically and over time, and how the use of such information in vaccine allocation decisions can improve the overall impact of vaccination. Information on the quantity of vaccines administered versus leftover in inventory in different locations could inform the decisions on how to allocate the next batch of vaccines geographically. Such visibility could be achieved in various ways, e.g., through vaccine registries.

Combining shipment and registry information would provide visibility into administered and leftover vaccine inventory. For example, the state of Oregon collects immunization data from both public and private health care providers to create vaccination records for individuals and reports immunization rates by county for the seasonal influenza vaccine. [44]. Ultimately, it is expected that the value of inventory visibility is most important when the vaccine supply is limited, which is often the case during an influenza pandemic. Vaccine inventory information, updated geographically and over time, could help reduce vaccine wastage while meeting the demand of the population in a fair and equitable manner, and reduce the disease burden by decreasing the number of infections.

3.2 Relevant Literature

Some of the literature on the allocation of limited vaccine supply focus on prioritizing certain sub-populations by age or other health risks and evaluating the benefits of targeting a limited vaccine supply [45, 46, 47, 48, 49], in line with the vaccine recommendations from the Advisory Committee for Immunization Practice (ACIP) [50] in the United States, where the risk groups may be specific to an influenza strain. Some researchers have quantified the benefit from the availability and allocation of vaccines early in a pandemic [51]. Prior work addressing the geographical allocation of a limited vaccine supply is scarce. Matrajt et al. propose a mathematical model to distribute vaccine in a network of cities in Southeast Asia connected by the airline transportation network; they find that a city-specific allocation strategy can reduce the attack rate substantially but at the expense of fairness [52]. Araz et al. consider the allocation of limited vaccine between and within the counties in the state of Arizona in the United States based on expected epidemic waves [53]. They find that a pro-rata strategy is effective when considering both the infection attack rate and the lead time for receiving vaccine inventory. Other authors consider shipping vaccines in two phases, where vaccines in the second phase may be sent to regions where the epidemic is not yet contained [54].

This paper proposes a modified pro-rata allocation strategy with respect to the demand for vaccine, by utilizing vaccine inventory information and allocating the available vaccine supply to any location where the individuals continue to request the vaccine. This is particularly important when uptake rates vary across population groups or geographical [55, 44, 56]. The proposed strategy thus maintains fairness with respect to the underlying demand from the population. Note that when the uptake rates are similar across geographical regions, the proposed strategy is equivalent to the traditional pro-rata (population based) strategy; however, the proposed strategy is more effective (in terms of reducing the number of infections and inventory waste, while maintaining fairness) when the uptake rates vary geographically.

3.3 Methods

3.3.1 Disease Simulation

We adapt a simulation-based disease spread model and use data from the state of Georgia in the United States with heterogeneous population mixing to predict the spread pattern of the disease both geographically and temporally. We use a detailed Susceptible-Exposed-Infected-Recovered (SEIR) model that tracks the disease status of an individual (we use one million simulated individuals, or agents, to represent the ten million population in the state of Georgia) as the disease spreads through a contact network by interactions in households, workplaces, schools, and communities. The model is flexible and can be run with data from other locations.

The method builds upon a previously-established agent-based simulation model [24, 57, 58]. Two main assumptions of the model are as follows (see Figure B.1 and B.2 in section B.1 for details):

- Every individual is in one of following stages at a given time: susceptible (S), exposed (E), pre-symptomatic (I_P), asymptomatic (I_A), symptomatic (I_S), hospitalized (H), recovered (R), or dead (D).

- The entire population has three levels of mixing: (i) community (day and night), (ii) peer groups (day), and (iii) household (night).

At the start of a simulation run, the entire population contact network is generated and every individual is susceptible. An initial virus is introduced randomly to infect individuals in selected census tracts. An infected individuals disease status changes to exposed (E). With pre-defined probabilities, the disease progresses within infected individuals and spreads to previously healthy individuals across the network. Once recovered (R) from the disease, the individual remains in that state.

To assess the effect of vaccination under different vaccine allocation strategies, we expand this simulation model by adding the option of vaccination; within fourteen days of vaccination, a person becomes immune to the disease (i.e., moves to the recovered state) with a positive probability [38].

The simulation outputs include the spatial and temporal estimates of the spread of the disease under different vaccine allocation strategies. The total infection attack rate (IAR) represents the cumulative percentage of the population who have been infected during the epidemic. The peak prevalence is the maximum percentage of the population infected at a given time.

An important parameter in the model is R_0, the reproductive number, which measures the transmission potential of the virus (i.e., the expected number of secondary infections caused by a typical infection). The analysis is presented for $R_0 = 1.5$; similar insights are obtained for $R_0 = 1.8$ and $R_0 = 2.0$.

3.3.2 Vaccine Allocation and Uptake

Since the capacity for the influenza vaccine is limited during a pandemic, vaccine supply becomes available in batches over time. Vaccine allocation begins in a vaccination start week and continues (e.g., on a weekly basis) until all the vaccine inventory is depleted or unmet demand reaches zero. Beginning with the vaccination start week, batches of vaccine arrive at each census tract in amounts that depend on the vaccine allocation strategy and the total vaccine availability.

The uptake rates often differ from one geographical location to another [56]. At the beginning of the simulation, we randomly select a subset of individuals (according to the uptake rate) in each census tract as willing to receive the vaccine. During each week, available vaccine is administered (randomly) to the individuals in that census tract who would like to be vaccinated, have not been infected previously, and are asymptomatic.

We consider two cases, where census tracts do and do not keep track of and report the remaining vaccine inventory levels on a weekly basis. When inventory levels are known geographically, they can inform the allocation strategy. The general principle is that areas with unused inventory could potentially receive less vaccine in the next allocation period.

We consider two strategies for allocating vaccine: (1) Population-Based (PB) (or pro-rata) strategy delivers available vaccine in each period proportional to the population size of each census tract. This is similar to the practice followed by many states during 2009-2010 [41]. (2) Population and Inventory-Based (PIB) strategy allocates vaccine (in proportion to the remaining unvaccinated population in each census tract) only to those census tracts that have zero inventory, i.e., those that already administered all the vaccine that was shipped earlier. PIB is motivated by a strategy used by manufacturers for allocating a limited supply of resources [59, 60]. When the uptake rates are equal in all census tracts, the PB and PIB strategies are equivalent. Detailed descriptions of both strategies ae presented in section B.2.

Each strategy is evaluated based on several criteria including the disease spread (e.g., IAR), operational aspects (e.g., vaccines shipped, administered, or leftover), and the service level (vaccine administered divided by the total number of susceptible individuals willing to receive the vaccine).

3.3.3 Experiments

We ran experiments simulating various scenarios with different parameters including vaccination start week (week four and seven), total vaccine supply (20%, 40%, 60%, and 80% of the population), time horizon over which the vaccine is delivered to census tracts (four, eight, and twelve weeks), and three different uptake rate settings: UTR_1: half of the census tracts have uptake rate 25% and the other half 75%. UTR_2: half of the census tracts have uptake rate 0% and the other half 100%. UTR_3: each census tracts uptake rate is randomly chosen from a uniform distribution between 0% and 100%.

We summarize all experimental parameters and provide justifications in Table B.1 (section B.3). To account for randomness, we generate five distinct contact networks and within each network we perform ten simulation runs (replications) for each uptake rate setting (see section B.4 for the analysis of the number of replications needed [61]). Hence, there are a total of $5 \times 10 = 50$ simulation runs for each of the $2 \times 3 \times 4 = 24$ parameter combinations across three uptake rate settings, resulting in a total of 72 scenarios and 3,600 simulation runs.

3.4 Results

We present results comparing the scenarios of no vaccination, and vaccine allocation under the PB and PIB strategies for $R_0 = 1.5$ and UTR_1, focusing on IAR and the percentage of leftover (i.e., allocated but not administered) vaccine inventory. Results on UTR_2 and UTR_3 are presented in section B.5. Unless stated otherwise, the results presented in this section are the average for the scenarios where the vaccination start week is four and the vaccine allocation horizon is eight weeks.

3.4.1 No Vaccination versus Vaccination under the PB Strategy

Without vaccination, the peak prevalence is 2.8%, which occurs around week ten, and the IAR is 50.1% (averaged over multiple simulation runs). The prevalence in every census tract is positive, and census tracts around the city of Atlanta have a higher IAR than rural census tracts.

When vaccine is available for 40% of the population and distributed under the PB strategy, the peak prevalence (1.2%) and IAR (23.4%) are lower and the peak occurs earlier compared to the no vaccination case (see section B.6 for further discussion). Vaccination under the PB strategy has a lower peak prevalence and IAR when vaccine is distributed earlier and over a shorter period.

Selected comparisons on peak prevalence and IAR under various parameters of vaccination are presented in section B.7. Detailed results for all combinations of vaccination start week, vaccine delivery horizon, and total available vaccine supply are presented in Table B.2 to B.10 in section B.5.

3.4.2 Vaccination under PB versus PIB Strategies

IAR relative to the total population under PIB is on average -0.1% to 1.2% lower than that under PB (Figure 3.1). These results, except for when the vaccine supply is 20% of the population, are significant under a two-sample t-test with a 95% confidence level. IAR under UTR_1 is generally lower than that under UTR_2 and UTR_3 for both PB and PIB, when all other parameters are the same. The average difference in IAR between PB and PIB (PB - PIB) ranges from 4.0% to 6.4% under UTR_2 and 0.9% to 1.6% under UTR_3.

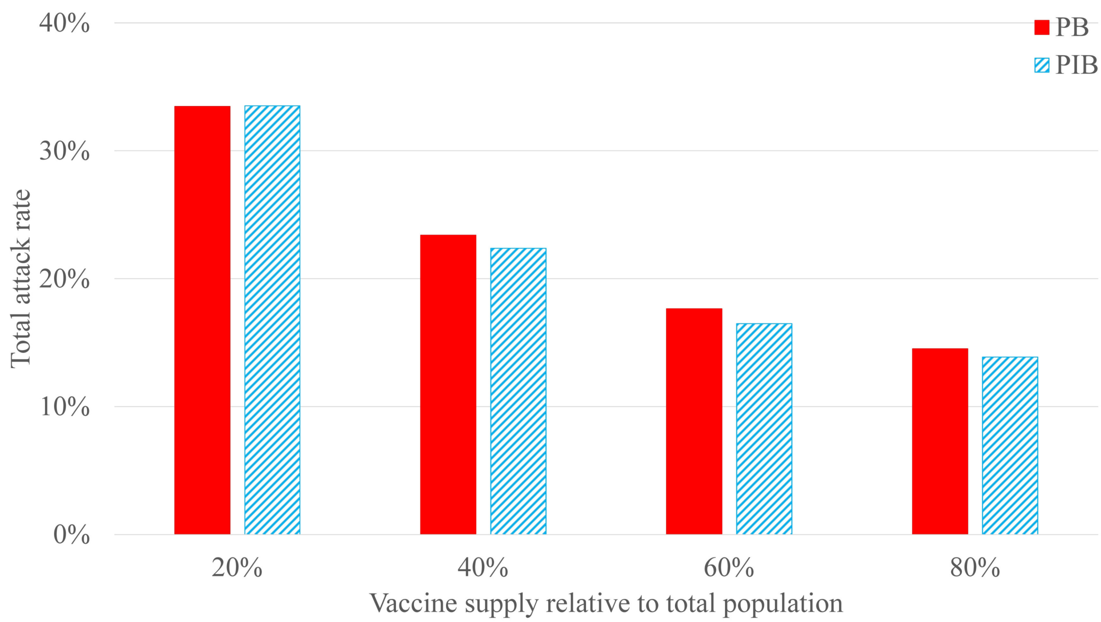

Figure 3.1: IAR under PB and PIB strategies.

Detailed comparisons in IAR and t-test results are shown in section B.5.

Additional results (presented in section B.6) indicate that an earlier vaccination start week and shorter vaccine distribution horizon reduce the IAR.

Figure 3.2 shows the number of vaccines shipped and administered under PB and PIB strategies. The solid, striped, and combined columns represent the vaccines administered, leftover, and shipped during the entire horizon, respectively. Under PIB, the total shipment is lower or the same, but the total amount of vaccine administered is higher compared to PB; hence, vaccine utilization (and service level) is higher, and the leftover inventory is lower under PIB. For example, when the vaccine supply is equal to 40% of the population, both PB and PIB ship all the available vaccine, but PIB administers 674 thousand more doses than PB. As a result, the leftover inventory is 20.7% of the total shipped under PB versus 3.8% of the total shipped under PIB.

The cost of the leftover vaccine inventory is estimated using information from a previous study in New York City [62, 63]. The (per dose) vaccine production cost is \$5.0 to \$10.0, distribution cost is \$1.5 to \$5.0, and disposal cost is \$0.1 to \$1.0. When the vaccine supply is sufficient to cover 40% of the population, PIB ships the same amount of vac-

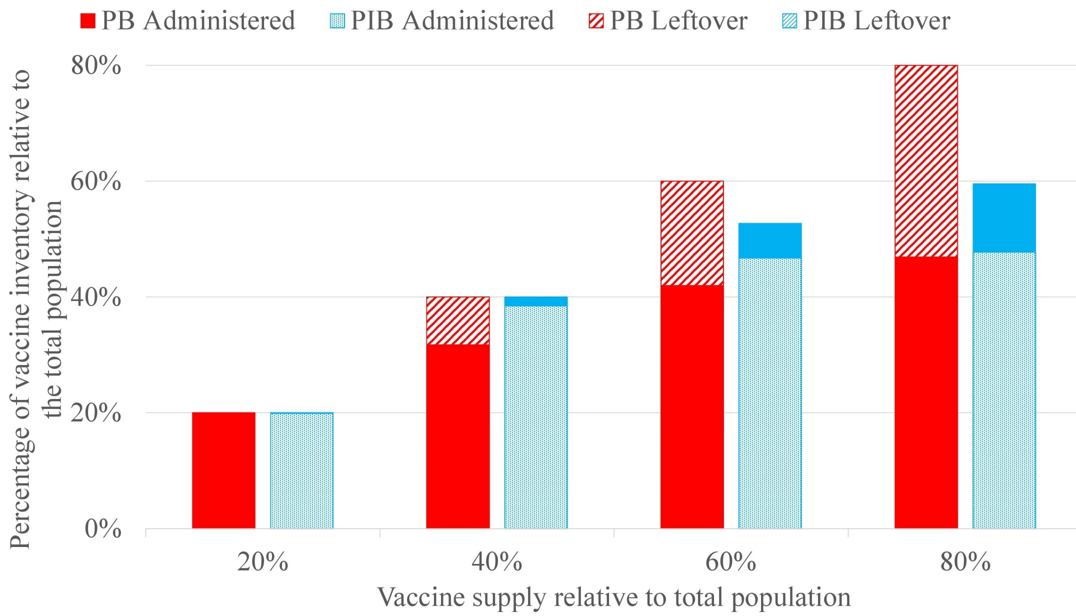

Figure 3.2: Vaccine shipped, administered, and leftover in inventory, as a percentage of the total population.

cine and results in 674 (95% CI on PIB-PB, 670 to 679, $p < 0.0001$) thousand less doses of leftover vaccine (or equivalently, more doses of administered vaccine) compared to PB (Figure 3.1). Hence, the estimated cost savings under PIB versus PB range from \$4.4 to \$10.9 million.

We also calculate the service levels under PB and PIB strategies. Demand refers to the number of people who were willing to receive the vaccine (calculated by multiplying the population in an area with the uptake rate), and Served refers to the number of people who were vaccinated. Service level is defined as Served/Demand $\times$ 100%. When the vaccine supply is sufficient to cover 40% of the population, the mean service levels under PB and PIB for census tracts with 75% uptake rate are 55.8% and 73.5% (95% CI on PIB-PB, 17.7% to 17.8%, $p < 0.0001$), respectively. For census tracts with 25% uptake rate, the average service level for PB and PIB are 97.8% and 97.9% (0.0% to 0.1%, $p = 0.0058$), respectively. Additional results and details are presented in section B.7.

3.5 Discussion

The results indicate that the PIB strategy dominates the PB strategy across multiple metrics. Given a fixed amount of vaccine supply, more vaccine is administered to the population (higher service levels), resulting in similar or lower IAR under PIB versus PB. Since PIB ships the vaccine (over time, as new batches become available) to those areas where there is still demand for the vaccine, versus shipping it to areas where the demand is saturated, it ships less vaccine, and hence, incurs a lower transportation cost and lower amount of leftover inventory, compared to PB. Note that the percentage of population vaccinated in any geographic area is higher under PIB versus PB; hence, the benefits of PIB are realized while maintaining fairness.

IAR under PIB is similar to or lower than that under PB. A 1.0% drop in IAR (with vaccine supply of 40% in Figure 3.2) implies approximately 100 thousand fewer influenza cases in the state of Georgia. In general, the reduction in IAR under PIB versus PB positively correlates with the variability in the uptake rates across locations (IAR reduction is the highest under UTR_2 and the lowest under UTR_1), i.e., the higher the variability in the uptake rates, the higher the benefits of PIB over PB. Additional results and discussions on the changes in IAR when the uptake rates are correlated geographically or when considering herd immunity can be found in section B.7.

Leftover (unused) vaccine inventory often incurs extra cost (including storage and disposal), as experienced during the last phase of the H1N1 influenza vaccine campaign [64, 65, 66]. These costs are even higher if the leftover vaccine is treated as a hazardous waste, as was required in some states for vaccine containing thimerosal. Inventory visibility enables the implementation of allocation strategies such as PIB, reducing the amount and the cost of leftover vaccine inventory, and the potential negative environmental impact.

Visibility in inventory has additional benefits that have not been explored in this study. For example, inventory information can be used to learn the uptake rate in each census tract, and states could design policies or information campaigns for areas with low uptake rates to create awareness, which could result in an increase in vaccination rates across the population and greater reductions in IAR. The lower number of infections along with the cost savings enabled by the PIB strategy would free up valuable resources which could be invested elsewhere to improve the availability and access to public health services.

Overall, this study suggests that visibility of inventory information in public health supply chains can have many benefits. Some states have adopted systems and practices to increase visibility in supply chains,7 and others may want to consider the potential benefits and costs of such practices. Ultimately, investments towards increasing the visibility in public health supply chains could increase effectiveness (reducing the disease burden) and efficiency (saving cost), while promoting equity (fairness).

3.6 Limitations

We assume that the uptake rates, i.e., the willingness to receive the vaccine, are constant over time (but we allow the uptake rates to vary by location). In practice, uptake rates may vary over time, e.g., they may be lower towards the end of the epidemic. However, since we limit the vaccine allocation horizon to several weeks around the peak (where one would expect the awareness about the epidemic to be high) it is reasonable to assume that the uptake rates would be somewhat stable during the vaccine allocation horizon.

The benefits of inventory visibility could be even higher when the uptake rates vary over time. For example, a decreasing trend in uptake rates might prompt a local government to increase efforts in generating public awareness regarding the benefits of vaccination, and conversely, and increasing trend could be communicated to the upper levels of the vaccine supply chain to ensure inventory availability.

In the models, we track the vaccine inventory in a semi-aggregate fashion for computational efficiency (e.g., vaccine inventory levels are computed for about 1,600 census tracts in the model vs. more than ten thousand providers in practice). There are well-known results in the supply chain management literature indicating that as the number of locations holding the inventory increases, stocking levels and costs increase (to meet a desired service level), or alternatively, service levels decrease for a given fixed amount of inventory. Hence, in practice, given a high number of inventory locations and potentially higher variability between uptake rates among different locations, we expect that the PIB strategy could be even more beneficial compared to the PB strategy.

CHAPTER 4

MULTI-PERIOD MATCHING UNDER RELAXED STABILITY

4.1 Introduction

Large-scale assignment and reassignment of workers to jobs with preferences is a common practice for many organizations. Making decisions over multiple periods allows organizations to balance desired and undesired assignments for workers or rotate workers through a set of functional roles. For example, the World Food Program (WFP) often rotate their workers among global locations; the U.S. Navy assigns more than 320,000 military officers to jobs of different functions annually [67]; and the National Resident Matching Program (NRMP) assigns more than 30,000 medical school graduates to residency in the U.S. each year [68].

In practice, workers are often asked to rank jobs in order of preference, and jobs rank workers based on their skill sets or according to their past performance. Similar to the generalized assignment problem, a utility function is usually applied for each worker-job pair to measure the quality of assignment. The utility function can depend on the position of job in worker's preference list, or the position of worker in job's preference lists, or both. For example, the min-max regret measure is commonly used to measure the quality of assignment [3, 4, 5, 6]. In the context of assigning jobs to workers, the quality of an assignment is often measured from the workers' perspective. The number of workers and jobs can be large (hundreds or thousands), while the number of periods can be relatively small (three or four years).

It can be important for organizations to keep workers happy by satisfying their preferences and providing assignments of good quality, which boosts morale and increases retention. One of the most commonly used framework for solving assignment problems with preferences is the Gale-Shapely stable matching model [2], in which the concept of stability is used to capture preferences and willingness to stay in two-sided markets. While satisfying preferences, organizations also need to ensure the quality of assignment for each worker, as there can be trade-offs between the two objectives.

In this research, we study assignments of workers to jobs over multiple periods in a two-sided market. The workers' and jobs' preference lists are complete, and there are no ties. To satisfy preferences, we introduce a definition of multi-period stability by aggregating the stability constraint in each period and relaxing the stability requirement. The objective function is to ensure the quality of assignments by maximizing the minimum individual utility of workers summed over all periods. We formulate an integer program, obtain bounds on the objective function value, develop heuristics to help organizations operationalize decisions, and provide insights on the trade-offs between stability and the quality of matchings. In addition, we explore settings where workers' preference lists change over time and provide modified formulation and heuristics. At last, we characterize the set of preference lists for which our heuristics provide a stable assignment.

Our research adds to the existing research, which mainly considers single-period and static settings or does not capture the trade-offs between stability and the quality of assignments. Section 4.2 provides additional context on the problem; section 4.3 presents an integer program model; section 4.4 through 4.7 provide the main results on analyzing the problem under different settings. We summarize all algorithms and heuristics in Section C.1, proofs of the theorems and corollaries in Section C.2, and pseudocodes in Section C.5.

4.2 Literature

The concept of stable matching has been applied in many other areas recently, but most research focuses on making decisions in a single-period. In the Gale-Shapely stable matching model, members of two disjoint groups (e.g., workers and jobs) provide preferences for being matched to members of the other group. A matching is stable when for any worker-job pair not currently matched, either the worker is assigned to a less preferred job, or the job is assigned to a less preferred worker, or both. In a non-stable matching, there exists a worker-job pair that blocks the current matching (called a blocking pair), in which both the worker and the job in the new match are individually better off. The Gale-Shapely model can be formulated as a linear program with a constraint requiring each worker-job pair to be stable [69, 70].

Some common applications of the stable matching problem include matching doctors to hospitals [71, 72], students to schools [73, 74, 75], kidneys to patients [76, 77], and tenants to apartments or houses [78]. Although in general there could be many stable matchings for a given preference list, many applications were studied under special preference lists relevant to our research, where a unique stable matching exists [79].

Most research on stable matchings also focuses on a static setting, although there are recent papers that address stability in a dynamic matching market. In these papers, stability over multiple periods takes into account the changing preferences or utility [80, 81], or agents from one side or both sides of the market arrive and exit over time [82, 83, 84]. Several researchers introduce different concepts of approximately stable matchings in a single period (e.g., a matching is stable as long as members in the current match would not be worse off by a certain measure) to find matchings with a higher social welfare [85, 86].

4.3 Problem definition

We consider assignments of a set of n workers, W, and a set of n jobs, J, over p periods. An assignment is represented as a one-to-one correspondence M of the set of workers and the set of jobs with n rows and n columns, where the entry (i, j) indicates the number of periods worker i is assigned to job j. We use the notation $[n]$ to denote the integer set $\{1, ..., n\}$.

We assume workers have provided a list of jobs ranked in order of preference and jobs also have a ranked ordering of workers. We also assume references are complete (worker either prefers j_1 over j_2 or j_2 over j_1) and transitive (if worker prefers job j_1 over job j_2 and prefers job j_2 over job j_3 then worker prefers job j_1 over job j_3). In addition, there are no ties in the preference lists.

Similar to the Gale-Shapely stable matching model, we study assignments of jobs and workers where stability is used for satisfying preferences. We assume that all worker and job preferences are known in advance and that the set of workers and jobs do not change over time, which is done similarly in [69, 70]. Thus we utilize a deterministic model for studying stable matchings in the multi-period setting. In the following sections, we define multi-period stability and objective function, and formulate the problem using an integer program.

4.3.1 Multi-period (relaxed) stability

The multi-period stability for a worker-job pair (i, j) is defined as the number of times the following condition is satisfied divided by the number of periods:

1. either worker i is assigned to a job better than job j,
2. or job j is assigned to a worker better than worker i, or
3. worker i is assigned to job j.

If the number of period is three and the above condition is satisfied two times for worker i and job j, then the stability of worker-job pair (i, j) is $\frac{2}{3}$. To measure the overall stability of the matching, we require that a minimum quantity $\alpha(0 \leq \alpha \leq 1)$ be met for all worker-job pairs. A multi-period matching is α-stable if the stability for each worker-job pair is at least α.

We choose this definition because it is consistent with existing research that formulates a single-period stable matching model as a linear program [69, 70]. When $\alpha = 0$, the stability constraint is redundant since every decision variable is non-negative. When $\alpha = 1$, we require each worker-job pair to be stable in all periods, which is equivalent to requiring a stable matching in each period. For a single period and $\alpha = 1$, the multi-period stability reduces to the stability constraint in [69].

Our definition aggregates individual stability constraints in each period and uses a parameter to indicate how much stability is required. The definition of multi-period stability requires each worker-job pair, instead of the entire matching, to be stable in a certain number of periods, which can be viewed as a natural extension of the definition of single-period stability. As a result, an α-stable matching can be achieved by combining a certain number of stable and non-stable single-period matchings, or by combining non-stable matchings only. Other studies have proposed alternative definitions that are more complex, emphasize on different characteristics of the multi-period setting, and often do not allow for different stability.

Although α can be any real number between 0 to 1, it suffices to consider a limited number of values, where αp is an integer. A single-period matching is either 0-stable (not stable), or 1-stable (stable). Similarly, the multi-period stability of any worker-job pair in a p-period matching can only take values from $\frac{i}{p}, i \in [p]$. Therefore, assignments to a p-period matching will provide the same "level" of stability unless α belongs to different intervals of $\left(\frac{i}{p}, \frac{i+1}{p}\right], i \in [p]$. For example, if $p = 3$, then it is sufficient to only consider α to be either $0, \frac{1}{3}, \frac{2}{3}$ or 1.

4.3.2 Objective function

Organizations may be interested in considering the utility for each worker-job pair, which depends on the position of job in the preference list of worker. Since we are interested in balancing the desired and undesired assignments across workers over multiple periods, we utilize an objective function that is similar to the min-max regret concept introduced in [3]. We further assume that every worker-job pair applies the same utility function according to the stated preference lists.

For each worker-job pair (i, j), we apply a utility function u_{ij}^k in each period k and maximize the minimum individual utility summed over all periods. The utility function u_{ij}^k is a function on the position of job j in the preference list of worker i. More specifically, for worker-job pairs (i_1, j_1) in period k_1 and (i_2, j_2) in period k_2, if job j_1 and j_2 are both ranked r^{th} in the preference lists of worker i_1 and i_2, respectively, then $u_{i_1j_1}^{k_1} = u_{i_2j_2}^{k_2} = f(r)$, where $f(r) : \mathbb{N}^+ \to \mathbb{N}^+ \cup \{0\}$ and f is non-increasing in r.

4.3.3 Formulation

We formulate the multi-period stable matching problem with a max-min utility objective, as outlined in Formulation 2. For all workers $i \in [n]$, jobs $j \in [n]$, and periods $k \in [p]$, we define the necessary notation, input, and decision variables in Table 4.1.

Table 4.1: Notation, input and decision variables in the multi-period stable matching formulation.

W, J	notation	the set of workers and jobs, respectively
n, p	notation	the number of workers (jobs) and periods, respectively
$p >_{W_i} j$	notation	worker i prefers job p to job j
$p >_{J_j} i$	notation	job j prefers worker p to worker i
α	input	the requirement on the multi-period stability
u_{ij}	input	the utility measure for worker i if assigned to job j
x_{ij}^k	decision variable	1 if worker i is assigned to job j in period k, 0 otherwise
s_{ij}^k	decision variable	1 if worker-job pair (i, j) is stable in period k, 0 otherwise
u_i	decision variable	the individual utility of worker i summed over p periods
U	decision variable	the minimum individual utility

We apply a standard formulation trick to convert the convex objective function into a linear objective function. The objective function (4.1), combined with constraints (4.6) and (4.7), maximizes the minimum utility of workers summed over all periods. Constraint (4.2) and (4.3) ensure that the assignment is a perfect matching (where every worker and job is matched), and constraint (4.4) and (4.5) are the multi-period stability constraints.

Formulation 2 Multi-period α-stable matching

$$\max \quad U \tag{4.1}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} x_{ij}^k = 1 \qquad \forall j \in [n], k \in [p] \tag{4.2}$$

$$\sum_{j=1}^{n} x_{ij}^k = 1 \qquad \forall i \in [n], k \in [n] \tag{4.3}$$

$$s_{i,j}^k \leq x_{ij}^k + \textstyle\sum_{p >_{W_i} j} x_{ip}^k + \sum_{p >_{J_j} i} x_{pj}^k \qquad \forall i, j \in [n], k \in [p] \tag{4.4}$$

$$\sum_{k=1}^{p} s_{ij}^k \geq \alpha p \qquad \forall i, j \in [n] \tag{4.5}$$

$$u_i = \sum_{k=1}^{p} \sum_{j=1}^{n} u_{ij} x_{ij}^k \qquad \forall i \in [n] \tag{4.6}$$

$$U \leq s_i \qquad \forall i \in [n] \tag{4.7}$$

$$x_{ij}^k, s_{ij}^k \in \{0, 1\} \qquad \forall i, j \in [n], \forall k \in [p] \tag{4.8}$$

Consider the multi-period stability constraints in equations (4.4) and (4.5). When $p = 1$ and $\alpha = 1$, these constraints reduce to the single-period stability constraint in the one-to-one stable marriage problem formulation [70]. When $p \geq 2$ and $\alpha = 1$, an optimal solution is to repeat the worker-optimal stable matching in all periods, since Formulation 2 only measures the utility for workers. Both problems can be solved in polynomial time via the Gale-Shapely algorithm. When α is 0, the multi-period stability constraints become redundant and can be removed from the formulation, which is still hard to solve.

There are several differences between Formulation 2 and the formulation in [70].

1. Formulation 2 has a convex objective function that maximizes the minimum utility over all workers, while the formulation in [70] has no utility assigned to each matching and the objective function is to maximize the number of matchings,

2. the right hand side of constraint (4.5) is α instead of 1 when $p = 1$, and

3. we only consider perfect matchings (every worker and job is assigned) because we assume complete preference lists.

Notice that α is a right-hand side parameter of Formulation 2, and the objective function value of the linear relaxation of Formulation 2 is concave on α. In other words, the objective function value decreases as α increases, and the rate of decrease is higher under larger alpha values. As greater stability is required, there is a larger impact on the quality of assignment from workers' perspective.

4.3.4 Vertical heterogeneity

The integer program we formulate in section 4.3.3 can be solved for any specific preference list for workers and jobs, and multiple optimal solutions can exist. However, there exists several preference lists of particular interest. In a paper by Eeckhout [79], the author describes the following two extreme cases of preference lists and their foundations in practical applications.

1. **Vertical heterogeneity**, where all jobs have the same preferences over workers (e.g., a strict ranking of army officers based on standard scoring systems or the test scores of students) and all workers have the same preferences over jobs (e.g., the location of the job or ranking of universities).

2. **Horizontal heterogeneity**, where all workers (jobs) have different preferences over jobs (workers) , but each worker (job) has a different most preferred job (worker) and in addition is the most preferred by that job (worker).

Workers with similar background or skills usually have preference lists that are similar to vertical heterogeneity, where workers and jobs have identical preference lists. On the other hand, when the background of workers are more diversified, their preference lists could be more similar to horizontal heterogeneity.

We are particularly interested in the vertical heterogeneity preference list, as similar preference lists arise naturally in practice. For example, the WFP have "hard" jobs (e.g., a job based in Afghanistan) that are generally not preferred by anyone, and "easy" jobs that are preferred by everyone (e.g., a job based in Rome, Italy). The preference lists of workers are expected to be similar, and they list the "easy" jobs at the top of their preference lists in each period. Under such preference structures, requiring assignments to be stable in each period may not always be desirable, because workers at the bottom of jobs' preference lists could be assigned to their least preferred jobs in all periods.

On the other hand, the problem is easy to solve under horizontal heterogeneity: the optimal solution is to assign workers their first choices in every period for any stability requirement. Clearly the objective function value is better under horizontal heterogeneity. Therefore, we focus on solving the multi-period stable matching problem under vertical heterogeneity, where relaxing stability improves the objective function value.

4.3.5 Other special preference lists

The vertical heterogeneity preference list also exhibit other interesting properties. More specifically, both vertical and horizontal heterogeneity preference lists are special cases of the Sequential Preference Conditions (SPC) preference lists, which are sufficient conditions for the uniqueness of a stable matching. The SPC is defined as follows in [79] and [87].

$$\forall \text{ worker } k \text{ and job } s, \text{ worker } k \text{ prefers job } k \text{ over job } s \text{ as long as } s > k. \tag{4.9}$$

$$\forall \text{ job } k \text{ and worker } s, \text{ job } k \text{ prefers worker } k \text{ over worker } s \text{ as long as } s > k. \tag{4.10}$$

In other words, workers (jobs) prefer jobs (workers) in the same rank order to jobs (workers) below the workers'(jobs') own rank order. For preference lists satisfying the above conditions, we can make a recursive argument beginning from assigning the highest ranked worker and job to each other, and any other matchings will be blocked by the identity matching (the matching where workers are matched to jobs with the same rank orders). As a result, the unique stable matching is to assign worker k to job $k, \forall k \in [n]$.

The uniqueness of stable matching is sometimes desirable for analysis of truth-telling in mechanism design. In a single-period setting with full stability required, workers and jobs can report preferences different from their actual preferences. Then, truth-telling by all workers and jobs is a Nash equilibrium if and only if there is a unique stable matching based upon actual preferences [87]. In addition, Ehlers [88] notices that the NRMP has very few stable matchings, implying that the submitted preferences could be close to satisfy the SPC conditions.

We primarily assume that preference lists of workers and jobs satisfy vertical heterogeneity and stay the same over time. In subsequent analysis, we consider dynamic preference lists with vertical heterogeneity in section 4.6. We also indicate under what other preference lists our results hold in section 4.7.

4.4 Analytical results under vertical heterogeneity

There are several characteristics of vertical heterogeneity and the preference lists staying the same that can be utilized in analyzing the problem in depth. Under these assumptions, we make observations that help us to obtain an upper bound on the objective function value. In addition, we study the structure of optimal solution using a greedy algorithm that determines the upper bound. Furthermore, we implement the integer program developed in section 4.3.3 under a special utility function. We show that the integer program can only provide solutions in reasonable amount of time for small size problems, emphasizing the need for developing fast heuristics.

4.4.1 Upper bound

We first make several observations on vertical heterogeneity that could help us to obtain an upper bound on the objective function value.

1. When the preference lists stay the same over time, we can simplify the notation of utility as $u_{ij}^k = u_{ij}, \forall k \in [p]$. When preference lists are identical, the unique stable

matching is the identity matching, and the utility function can be further simplified as $u_{ij} = u_j, \forall i \in [n]$.

2. If $\alpha = 1$, worker k will be assigned to job k in all periods $\forall k \in [n]$, giving an objective function value $p \cdot u_n$. This is the minimum value that the objective function can achieve under any preference list.

3. The sum of total utility stays the same regardless of assignments.

$$\sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij}^k x_{ij}^k = \sum_{j=1}^{n} u_j \sum_{k=1}^{p} \sum_{i=1}^{n} x_{ij}^k = \sum_{j=1}^{n} u_j \sum_{k=1}^{p} 1 = p \sum_{j=1}^{n} u_j. \tag{4.11}$$

We exploit these properties to find an upper bound on the optimal objective function value, where the bound depends on α.

Theorem 1. *Given $n \geq 2, p \geq 2$ and $\alpha \in (0, 1)$, an upper bound on the optimal solution can be found by a greedy algorithm, where the upper bound depends on n, p and α.*

We design Algorithm 1 that iteratively assigns jobs to a subset of workers in order of their ranking orders in jobs' preference lists. This algorithm exploits observation 3 (equation 4.11) and satisfies stability requirements for each worker-job pair using minimal utility.

In each iteration, Algorithm 1 assigns one worker αp times to his or her unique stable matches, then assigns jobs with the least utility to the current worker in the other periods to maximize what is left for the unassigned workers. At the end of each iteration, we calculate the *average remaining utility* by taking the total utility of unassigned jobs and divide it by the number of unassigned workers, and we compare the individual utility of the current worker and the average remaining utility of unassigned workers. The stopping criteria is when the current worker has a lower utility than the average remaining utility.

We show that the final average remaining utility is an upper bound on the objective function value and the maximum number of iterations of Algorithm 1 is at most n. The number of workers assigned in Algorithm 1 depends on α. If $\alpha = 0$, no workers is assigned; if $\alpha = 1$, all workers are assigned to their unique stable matching in all p periods. In general, more workers are assigned by Algorithm 1 for larger alpha values.

The structure of the assignments from Algorithm 1 also provides insight on the structure of optimal solutions. Suppose for a given problem with α stability, the number of workers assigned by Algorithm 1 is k^*. We show in Theorem 2 that the assignments of the first k^* workers in any optimal solution share the same structure.

Theorem 2. *For any $n \geq 2, p \geq 2$ and $\alpha \in (0, 1)$, an optimal assignment M can be transformed into another optimal solution M' such that workers $k, k \in [k^*]$ are assigned the same as in Algorithm 1. The transformation algorithm terminates in $O(n^2 p)$ steps.*

We describe this algorithm in details in section C.1. The intuition of Algorithm 2 comes from that Algorithm 1 assigns the first k^* workers using jobs of minimal utility, so any other optimal solution with a different assignment will only provide less utility for the remaining workers, leaving the potential of improvement on the objective function value. Algorithm 2 identifies workers among the first k^* with excessive utilities and switches jobs with high utility to the rest of workers while maintaining stability for all pairs. Using these results, we prove that the multi-period stable matching problem is $\mathcal{NP}$-hard when preference lists are identical under mild assumptions on the utility function (see section C.2.3 for details). We also show that there exist special cases for which the problem is solved in polynomial time.

4.4.2 Reverse rank utility function and integer program performance

The min-max regret objective is widely used as an outcome measure in other research of stable matchings in a single period [89, 5, 86]. When worker i is matched with job j, the regret of worker i is defined as the position (or rankings) of job j in the preference list of worker i, and vise versa. The regret of the matching is the maximum regret over all workers and jobs and the max-min regret objective balances the quality of matching from the perspective of both workers and jobs. In our research, we define the utility u_k of job k, similar to regret, as the reverse rank of job k on workers' preference list: $u_k = n - k + 1, \forall k \in [n]$. We assume reverse rank utility for the rest of this chapter.

The computational time of the integer program varies with the number of workers and jobs, the stability required, and the structure of the preference lists. We implement Formulation 2 under vertical heterogeneity and reverse rank utility in CPLEX. The problem can be easy to solve under certain assumptions, such as when $\alpha = 1$. However, for $n = 30, p = 3, \alpha = \frac{2}{3}$, and the preference lists are vertical heterogeneous, the problem took more than six hours to find an optimal solution. Furthermore when the number of workers and jobs are more than 100, the integer program usually does not find a solution within 10% of the optimal value under one hour.

As a result of the integer program performance, we turn towards other algorithms and heuristics for finding solutions within reasonable time. The analytical results also allow us to make certain guarantees on the performance of the algorithms and heuristics.

4.5 Heuristics under vertical heterogeneity and reverse rank utility

4.5.1 Properties when stability is low (approximately $\frac{1}{2}$ or less)

For most practical problems, it may be desirable to have larger alpha values in order to satisfy preference. However, we first consider the solution and its structure for smaller alpha values, as this provides a building block for solving problems under larger alpha values.

More specifically, we focus on when $\alpha \leq \frac{\lfloor p/2 \rfloor}{p}$. Notice that for even number of periods, $\frac{\lfloor p/2 \rfloor}{p} = \frac{1}{2}$; for odd number of periods, requiring $\alpha \leq \frac{1}{2}$ is equivalent to requiring $\alpha \leq \frac{\lfloor p/2 \rfloor}{p}$. We obtain an optimal solution for $\alpha = \frac{\lfloor p/2 \rfloor}{p}$ and show that the optimal value does not change when α decreases from $\frac{\lfloor p/2 \rfloor}{p}$ to 0. In other words, we get $\alpha = \frac{\lfloor p/2 \rfloor}{p}$ "for free". The solution approach is summarized as Algorithm 3.

Theorem 3. *Under vertical heterogeneity, $\alpha \leq \frac{1}{2}$ and the utility of job k is $u_k = n - k + 1, \forall k \in [n]$, the optimal solution to the multi-period stable matching problem can be obtained in $O(n)$ steps.*

To understand what is driving the objective function value, we first consider the case when $\alpha = 0$. For any assignment, the stability constraints are satisfied for all worker-job pairs and the problem reduces to maximizing the minimum worker utility over multiple periods. Since workers have the same preference lists in all periods, job k will always provide a utility of $n - k + 1$ regardless of the assignment. Therefore, the total utility stays the same and is equal to:

$$p \sum_{k=1}^{n} n - k + 1 = \frac{pn(n+1)}{2}. \tag{4.12}$$

To maximize the minimum worker utility, ideally every worker should have the same utility $\frac{p(n+1)}{2}$. This objective value can be achieved easily when the number of periods is even: for all workers $k \in [n]$, assign worker k to job k in half of the periods and to job $n - k + 1$ in the other half. The total utility gained for any worker k in this assignment is:

$$\frac{p}{2}(n - k + 1) + \frac{p}{2}(n - (n - k + 1) + 1) = \frac{p(n+1)}{2}. \tag{4.13}$$

Since in the unique stable matching, job k is assigned to worker k for all $k \in [n]$, the stability of this solution is $\frac{1}{2}$. The details for when the number of periods is odd can be found in section C.1.

4.5.2 Properties when stability is high (approximately $\frac{1}{2}$ or more)

For higher values of stability, we design different heuristics that exploit the structure of the problem. Recall that it is sufficient to analyze alpha values with αp integral. Therefore for $\frac{1}{2} < \alpha \leq \frac{3}{4}$, we can assume $p \geq 3$.

We design Heuristic 1 with assignments in αp periods responsible for satisfying the required stability and in the other $(1-\alpha)p$ periods improving the objective value. From Algorithm 1, the stability constraints are the most restrictive for workers that are "popular" among jobs. As a result, the objective function values are often driven by workers that are "unpopular" among jobs. We design heuristics to first satisfy the stability of pairs involving "popular" workers using minimal utility, then improve the individual utility of "unpopular" workers in other periods. More specifically, we divide workers into two sets (the number of jobs in each set is $\lceil \frac{n}{2} \rceil$ for $\alpha \leq \frac{3}{4}$ and greater than $\lceil \frac{n}{2} \rceil$ for $\alpha > \frac{3}{4}$) and provide different assignment approaches for workers in each set.

Theorem 4. *Under vertical heterogeneity for any $p \geq 3$ and $\frac{1}{2} < \alpha \leq \frac{3}{4}$, Heuristic 1 gives an α-stable assignment.*

In Heuristic 1, the first half of workers $k \leq \lceil \frac{n}{2} \rceil$ are assigned to their unique stable matches. The rest of workers are be assigned to jobs $j \leq \lceil \frac{n}{2} \rceil$ in the remaining $(1-\alpha)p$ periods, which contributes to their stability since workers prefer jobs $j \leq \lceil \frac{n}{2} \rceil$ over jobs $j > \lceil \frac{n}{2} \rceil$. However, we still need an additional $(\alpha - (1-\alpha))p = (2\alpha - 1)p$ periods of assignments to guarantee stability for pairs involving workers $k > \lceil \frac{n}{2} \rceil$. We assign worker k to job k in $(2\alpha - 1)p$ of the periods, and in the rest of periods we improve the utilities among workers $k > \lceil \frac{n}{2} \rceil$ using similar procedures as in Algorithm 3.

We develop a constructive heuristic (Heuristic 2) for $\alpha > \frac{3}{4}$ similar to Heuristic 1, except that the number of workers assigned similarly to the first half of workers in Heuristic 1 is now greater than $\lceil \frac{n}{2} \rceil$ and depends on α. We also develop Heuristic 3 that improves the objective function value iteratively, although without theoretical performance guarantees.

4.5.3 Heuristic performance

To measure the theoretical performance of Heuristic 1 and 2, and the computational performance of all heuristics, we calculate the upper bound on the objective value for any given n, α, and p using Algorithm 1. We also obtain a closed form upper bound when n is sufficiently large (≥ 100).

Theorem 5. *If $n \geq 2, p \geq 3, \alpha > \frac{1}{2}$, and the utility of job k is the reverse rank $u_k = n - k + 1$, then the upper bound on the optimal value as the number of agents n goes to infinity is:*

$$UB(\alpha) = (1-\alpha)^2 + \sqrt{\alpha(1-\alpha)(3\alpha - \alpha^2 - 1)}. \tag{4.14}$$

Figure 4.1 shows the computation of $UB(\alpha)$ from Theorem 5. The upper bound $UB(\alpha)$ decreases as α increases, and is concave in α when n goes to infinity. This result is consistent with our analysis on the objective function value of the linear relaxation of the problem. The actual optimal value may not have the same shape as the decision variables are discrete.

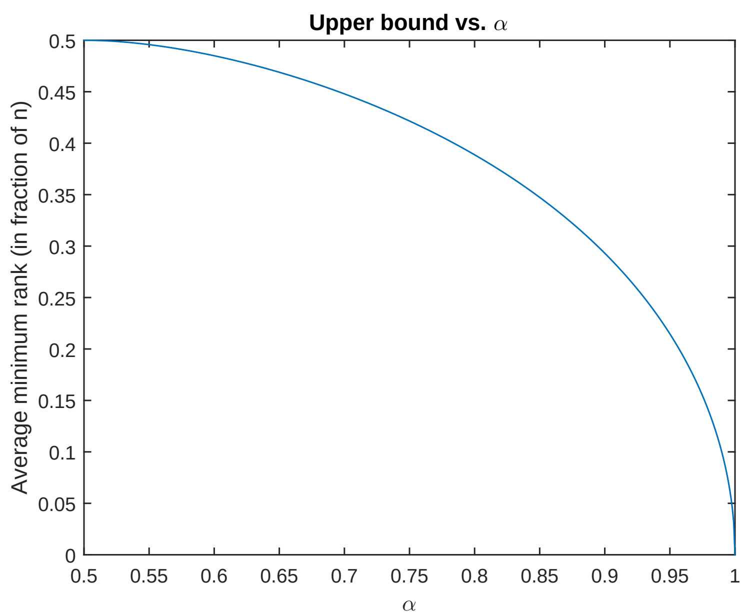

Figure 4.1: Upper bound value for each α (normalized by the number of workers and jobs n and the number of periods p)

With calculations of the upper bound, we quantify the theoretical performance of Heuristic 1 in Corollary 6 and Heuristic 2 in Theorem 7.

Corollary 6 (Performance of Heuristic 1)**.** *With sufficiently large n (≥ 100), for any $p \geq 3$ and $\frac{1}{2} < \alpha \leq \frac{3}{4}$, Heuristic 1 gives a solution that is at least 88% optimal.*

Theorem 7 (Performance of Heuristic 2)**.** *Let $u_{stm}(\alpha)$ be the minimum utility among all workers in Heuristic 2 for a given α, normalized by $n(\geq 100)$ and p, then*

$$u_{stm}(\alpha) = \min\left\{UB(\alpha) - \frac{1}{2p}, \frac{1-\alpha}{UB(\alpha) + 1/(2p) + 1 - \alpha}\right\}. \tag{4.15}$$

The heuristic gives a solution that is at least 73% optimal.

The worst-case theoretical and computational performance of Heuristics 1 and 2 are shown in Figure 4.2 when the number of workers and jobs ranges from 990 to 1010, the number of periods ranges from 3 to 50, and for every α where αp is an integer. The worst-case computational performance for Heuristic 3 is shown in Figure 4.3.

The theoretical and computational gaps are similar for Heuristic 1, but are different for Heuristic 2. The difference is bigger when the number of periods is large (≥ 10) and smaller when the number of periods is small. Notice that when the number of period is sufficiently large, the optimal solution can be approximated by solving its linear relaxation. This could explain why the performance of Heuristic 2 is better when the number of periods is larger for the same α.

Similarly for Heuristic 3, the computational performance is generally better when the number of periods is larger, but the optimality gap never exceeds 3%. Although Heuristic 3 gives much better computational performance under certain cases, the computation time is longer. In addition, Heuristic 3 only applies when workers and jobs have identical preference lists, while at the end of this paper we are able to generalize the other two heuristics and apply to other preference lists.

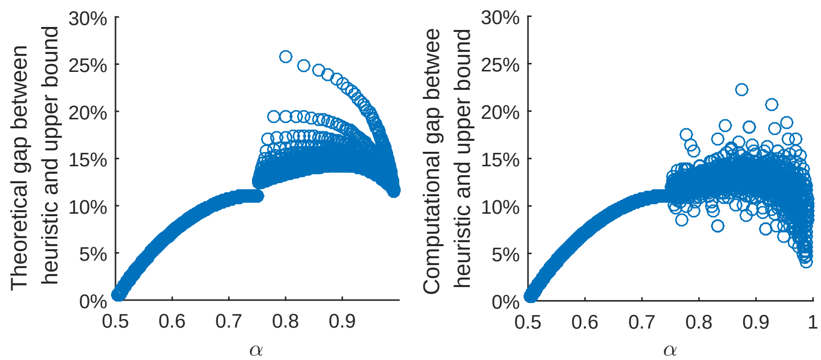

Figure 4.2: Combined performance of Heuristic 1 and 2 compared to upper bound

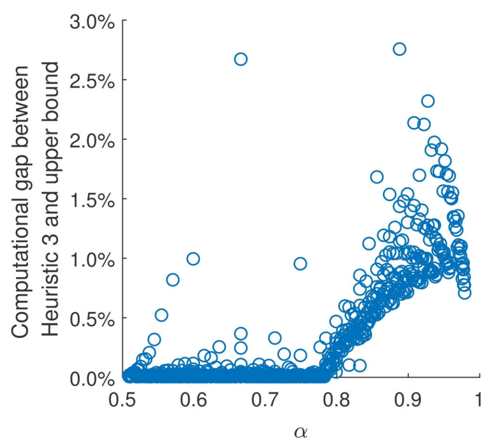

Figure 4.3: Computational performance of Heuristic 3.

4.6 Workers' preference lists change over time

An important setting not captured by Formulation 2 is when worker preferences change over time. Certain companies (General Electric [90]), organizations (The World Food Program) and the NRMP establish job rotation programs, in which workers, employees, or doctors have the incentive to change their preference lists over time. These rotation programs are usually limited to a few years as the ability of individuals to think over many time periods is limited from the well known concept of bounded rationality [91]. Studies that address preference list dynamics (although without studying the trade-offs between stability and quality of matchings) usually do not solve the problem in full generality [80, 81].

We consider a special case of a generally changing preference list, driven by cases when jobs can be used to acquire skill sets. We refer the problem when preference lists change over time "the dynamic problem" and the case when workers' preference lists stay the same over time "the static problem". We assume vertical heterogeneity on the initial preference lists and the dynamic of preference lists is the same for all workers. All other assumptions for the static problem hold true for the dynamic problem.

4.6.1 Dynamic problem definition

We focus on the case when all workers prefer jobs they have not been assigned over jobs they have already been assigned. Let the number of workers and jobs be n and the number of periods be p. We summarize the dynamic of preference lists in each period for a worker as follows.

1. In period one, workers are given a specific preference list (identical preference lists). Let the current period be $q = 1$.

2. If worker k is assigned to job s in period q, then in period $q + 1$ job s becomes the least preferred job of worker k. The relative rank orders of jobs other than s in worker k's preference list in period $q + 1$ is the same as in period q.

3. In any period, the utility of a job with rank order k is $n - k + 1$ (reverse rank utility).

Although the preference lists change in the same way for all workers, they will be different for each worker at the end of each period. As a result, whether a worker-job pair becomes a blocking pair in a specific period does not depend on assignments in other periods. Therefore, the assignment matrix representation in the static problem is no longer adequate as there is ambiguity on whether pairs are blocking a matching or not. We introduce another assignment matrix N with p rows (periods) and n columns (jobs), and the (i, j) entry of N indicating the index of job worker j is assigned to in period i.

Many of the results in this section extend to the case when an entire group of jobs moves to the bottom of the preference lists after workers are assigned to only one job within the group. See section C.4 for further discussion.

4.6.2 Dynamic problem formulation

The dynamic problem can be easy to solve when the number of periods is one or two. When $p = 1$, the dynamic problem is equivalent to the static problem and any solution is optimal. When $p = 2$, an optimal solution is to assign the worker-optimal matching in period one, then assign jobs with higher utility to workers with lower utility in period two.

We formulate an integer program for the three-period dynamic problem using exponential number of if-then constraints. The detailed description of the formulation can be found in section C.3. We implement the dynamic formulation in CPLEX and are able to solve the problem with $n \leq 9, p = 3$ and $\alpha = \frac{2}{3}$ to optimality under one hour. For problems with more than 20 workers and jobs, the integer program usually does not find a solution within 10% of the optimal value.

It is possible to formulate an integer program for problems with more periods, but the number of variables and constraints needed is extremely large. We focus our study on three periods, as dynamic problems with more periods provide limited (additional) insights for understanding the structure of the dynamic problem.

4.6.3 Heuristics for the dynamic problem

Since the dynamic integer program formulation does not provide solutions under reasonable time, we solve the three-period dynamic problem by modifying Algorithm 3 for $\alpha = \frac{1}{3}$ and Heuristic 1 for $\alpha = \frac{2}{3}$. To account for the changes in workers' preference lists, we specify the assignments in each period (e.g., "assign worker k to job k in period one and two" instead of ”assign worker k to job k” two times). Since both Algorithm 3 and Heuristic 1 assign jobs in either descending or ascending order of the index of workers during each step, the only modification needed is to update workers' preference lists after each period.

In section C.1 we outline Heuristic 4 for $\alpha \leq \frac{1}{3}$ and Heursitic 5 for $\frac{1}{3} < \alpha \leq \frac{2}{3}$. The modified heuristics could be used to solve for when $\alpha \leq \frac{3}{4}$ and $p \geq 4$ under minor changes, but not when $\alpha > \frac{3}{4}$ as Heuristic 1 applies only to when $\alpha \leq \frac{3}{4}$. In this case, we conjecture that similar heuristics could be developed by further modifying Heuristic 2.

4.6.4 Dynamic problem computational performance

To measure the performance of the dynamic heuristics, we find an upper bound on the objective function value that does not depend on α by utilizing the structure of dynamic preference lists. In addition, we implement heuristics for the dynamic problem and compare their performance with the upper bound for the dynamic problem.

Jobs with lower utility in period one could provide more utility in later periods, especially when assigned to workers who have already been assigned to higher utility jobs in previous periods. When workers have identical preference lists, we obtain an upper bound on the optimal value by computing the *maximum potential utility* of each job in each period and taking the average maximum potential utility over all periods and jobs.

Theorem 8. *Let $U_{max}(j,t)$ be the maximum potential utility of job j in period t, then*

$$U_{max}(j,t) = \min\{n, n-j+t\}, \tag{4.16}$$

and there exists a ”rotating assignment” that uniquely achieves the following upper bound:

$$UB_{dy}(n,p) = \left\lfloor \frac{1}{n} \sum_{j=1}^{n} \sum_{t=1}^{p} U_{max}(j,t) \right\rfloor. \tag{4.17}$$

We expect the upper bound UB_{dy} to be larger than the optimal objective function value in general. This is more likely to be true when $\alpha > \frac{1}{2}$ because for the static problem, the upper bound decreases as α increases beyond $\frac{1}{2}$.

Using UB_{dy} as an upper bound, we analyze the computational performance of heuristics for the dynamic problem. For n from 1 to 100 and $p = 3$, we compute the objective values of Heuristic 4 with $\alpha = \frac{1}{3}$ and Heuristic 5 with $\alpha = \frac{2}{3}$, and compare these results to the dynamic problem upper bound UB_{dy} and the objective value of the static Heuristic 1 with $\alpha = \frac{2}{3}$ in Figure 4.4. We make several observations from Figure 4.4.

1. In both heuristics, the objective function value increases roughly linearly with the number of workers and jobs n, as well as the upper bound. Furthermore, the optimality gap of the heuristics appears to be converging to a constant as the number of workers and jobs grows.

2. The performance of the dynamic heuristic for $\alpha = \frac{1}{3}$ (Heuristic 4) is close to the dynamic upper bound. In fact, the absolute gap between the heuristic performance and the upper bound is a constant 2. We expect the absolute gap stays constant when the number of jobs increases, and the percentage gap converges to 0 as $n \to \infty$.

3. The dynamic heuristic for $\alpha = \frac{2}{3}$ (Heuristic 5) returns a slightly higher objective function value compared to the static heuristic for $\alpha = \frac{2}{3}$ (Heuristic 1), which is expected because jobs in the dynamic problem generally provides more utility in later periods. The percentage gap between the heuristic performance and the dynamic upper bound is 17%. For the static problem, the theoretical optimality gap is 10% because the static upper bound depends on α. It is reasonable to expect the actual optimal value to decrease as α increases for the dynamic problem. Therefore, the heuristic performance gap could be smaller than appears.

In our computational results, the number of periods is small compared to the number of workers and jobs. As a result, the relative difference between the workers' preference lists in the static and dynamic problems is expected to be small. The optimality gap is expected

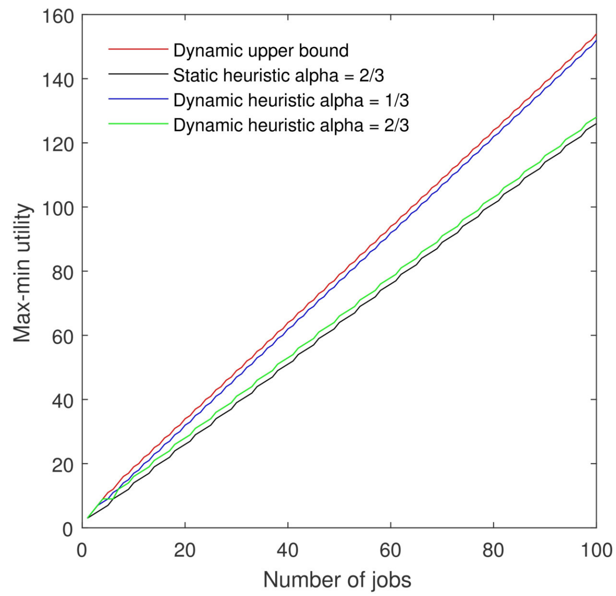

Figure 4.4: Computational results for Heuristic 4 and 5.

to grow as the number of periods increase, because the dynamic upper bound does not depend on α.

4.7 General preference lists

Vertical heterogeneity is an extreme case of preference lists and in practice the preference lists of workers and jobs may not be identical. In this section we extend our analysis to a more general set of preference lists, where a unique stable matching exists in a single period problem. For the static and dynamic problems, we identify conditions under which the developed heuristics will return an α-stable solution. We provide insights on the necessity of these conditions and relate them to the literature. In addition, we comment on the computational performance of the heuristics under these preference lists. We only discuss the case when $\alpha > \frac{1}{2}$ as in practice higher stability is desired, and leave the discussion of other cases in the Appendix.

4.7.1 The static problem

We focus on the case when $\frac{1}{2} < \alpha \leq \frac{3}{4}$ and analyze the proof of Heuristic 1, which uses conditions that are less restrictive than vertical heterogeneity to show each worker-job pair achieves α-stability. We summarize the conditions and their implications for Heuristic 1 to return an α-stable assignment below.

- For $k \leq \lceil \frac{n}{2} \rceil$, worker k is assigned to job k at least αp times
  1. For $s \leq k$, job s prefers worker s to worker k: the stability of pair (k, s) is greater than α.
  2. For $s > k$, all workers prefer jobs with higher utility than s to jobs with lower utilities than k: the stability of pair (k, s) is greater than α.
- For $k > \lceil \frac{n}{2} \rceil$, worker k is assigned $(1 - \alpha)p$ times to job $s \leq \lceil \frac{n}{2} \rceil$.
  1. For $s > \lceil \frac{n}{2} \rceil \wedge s \leq k$, job s is assigned to worker s at least $(2\alpha - 1)p$ times. Therefore the stability of pair (k, s) is $(1 - \alpha) + (2\alpha - 1) = \alpha$.
  2. For $s > k$, worker k is assigned to at least $(1 - \alpha + 2\alpha - 1)p = \alpha p$ jobs that have higher utilities than job s. Therefore, the stable-level of pair (k, s) is at least α.

We notice that vertical heterogeneity is a special case to the above conditions used in the proof of Theorem 4. As a result, Heuristic 1 can return an α-stable solution for a broader set of preference lists, which is a subset of the SPC conditions. We summarize these conditions in Corollary 9.

Corollary 9. *Heuristic 1 returns an α-stable solution if the following conditions on workers' and jobs' preference lists hold:*

1. *$\forall k \in [n]$, worker k prefers job k to all jobs $s, s > k$,*

2. $\forall k \in [n]$, *job* k *prefers worker* k *to all workers* $s, s > k$, *and*

3. $\forall k > \lceil \frac{n}{2} \rceil$, *worker* k *prefers all jobs* $s, 1 \leq s \leq \lceil \frac{n}{2} \rceil$ *to all jobs* t, $\lceil \frac{n}{2} \rceil + 1 \leq t \leq n$.

Proof. Immediately from the proof of Theorem 4. □

Conditions 1 and 2 in Corollary 9 are exactly the sufficient conditions SPC. This is not surprising since Heuristic 1 is partially symmetric along the identical matching, which represents the unique stable assignment under SPC. These assignments are responsible for satisfying stability for many worker-job pairs. Condition 3 requires workers with lower rank orders prefer jobs with higher rank orders to jobs with lower rank orders, with job $\lceil \frac{n}{2} \rceil$ separating the two sets. Notice that requiring the preference lists of the second half of workers to be identical is a special case of condition 3, which reflects the situation where half of the jobs is clearly better than the other half in workers' preference lists.

The set of preference lists for which Heuristic 2 returns a stable matching is similar to the conditions in Corollary 9, replacing the third condition by: $\forall k > n - x$, worker k prefers all jobs $s, 1 \leq s \leq n - x$ to all jobs $t, n - x + 1 \leq t \leq n$. Notice that this condition depends on the quantity $x(x < n - \lceil \frac{n}{2} \rceil)$, which decreases as α increases.

We also observe that for arbitrary job preference lists, we can obtain condition 2 by reordering the workers' indexes. Suppose workers and jobs are indexed from 1 to n. We define the favorite worker of job 1 as worker w_1, and subsequently define worker w_k as the favorite worker of job k, excluding workers w_1 to w_{k-1}. We then reorder the workers from w_1 to w_n and reflect the new ordering in the preferences lists of jobs. Workers' preference lists stay the same, but the resulting jobs' preference lists now satisfy condition 2. A similar argument can be made for arbitrary worker preference lists, where we obtain condition 1 by reordering the jobs' indexes.

4.7.2 The dynamic problem

We focus on the case when $\frac{1}{3} < \alpha \leq \frac{2}{3}$ for a three periods problem and analyze the set of preference lists under which heuristics for the dynamic problem returns α-stable assignments. Results for the dynamic problem are similar, but the set of preference lists we can generalize is more restricted. More specifically, Heuristic 5 returns an α-stable solutions for a broader set of preference lists related to the SPC condition, specified in Corollary 10.

Corollary 10. *Heuristic 5 provides an α-stable assignment if the following conditions on workers' and jobs' preference lists hold:*

1. $\forall k > \lceil \frac{n}{2} \rceil$, *worker k prefers all jobs $s, 1 \leq s \leq \lceil \frac{n}{2} \rceil$ to all jobs t, $\lceil \frac{n}{2} \rceil + 1 \leq t \leq n$.*
2. $\forall k \in [\lceil \frac{n}{2} \rceil]$ odd, *worker k prefers job $k+1$ to all jobs $s, s > k+1$ and $\forall k \in [\lceil \frac{n}{2} \rceil]$ $even$, worker k prefers job $k-1$ to all jobs $s, s > k$, and*
3. $\forall k \in [\lceil \frac{n}{2} \rceil]$ odd, *job k prefers worker $k+1$ to all workers $s, s > k+1$ and $\forall k \in [\lceil \frac{n}{2} \rceil]$ $even$, job k prefers worker $k-1$ to all workers $s, s > k$.*

In addition, conditions (4.9) and (4.10) hold.

Proof. Immediately from the proof of Theorem 14. □

Condition 1 is the same as condition 3 in Corollary 9, which is expected as both of the heuristics put an effort to improve the utility of workers with lower ranked orders. Condition 2 and 3 are the results of avoiding identical assignments in the heuristics. For example, if worker 1 is assigned to job 1 in period one, then by condition 2 worker 1 is assigned to job 2 in period two. Without this condition, worker 1 would not necessarily prefer job 2 to all other jobs in period two, which is essential to maintain stability for all pairs $(1, k), k \geq 3$ in period two. Similarly for job 1, condition 3 maintains the stability for all pairs $(k, 1), k \geq 3$.

To solve the dynamic problem with higher number of periods and larger α, we expect that making similar modifications to the static heuristics is sufficient. However, as the number of periods grows, workers' preference lists will be less likely to satisfy the SPC conditions and more conditions will be needed (eventually the preference lists all converge to vertical heterogeneity as the number of periods increases).

In the proofs of Algorithm 3 and Heuristic 1, 2, 4 and 5, we prove that the objective function value returned under each heuristic can be only higher compared to when workers have identical preference lists. Therefore, we expect the computational results be better under these more general preference lists.

4.8 Conclusions

We study the multi-period assignment problem under complete preference lists, no ties, and relaxed stability. Our paper contributes to the discussion of assigning jobs to workers over time, which is part of the regular operations for many organizations such as the World Food Program, Navy, and the National Residency Matching Program. We address the problem in a two-sided market under complete preference lists and no ties, where the preference of workers either stay the same or change over time.

In particular, we extend stability to a multi-period setting where organizations need to make assignments that are stable and of good quality, for which existing literature often does not simultaneously account. We formulate an integer program model and develop heuristics that are fast and of good computational performance, and we provide insights on the trade-offs between requiring assignments with higher stability and higher quality.

In the first part of our paper, we study the theoretical aspects of the problem under a special preference list vertical heterogeneity. We analyze the structure of optimal solutions and provide an upper bound on the objective value using a greedy procedure. We exploit the structure of the assignment from this greedy procedure and show that a special case of the multi-period stable matching problem is equivalent to the well known partition problem.

In the second part of our paper, we provide constructive heuristics that are fast and easy to implement. We solve the problem with large number of agents and small number of periods using these heuristics. When workers' preference lists stay the same over time and are identical, the best theoretical performance is within 27% of the optimal value and the best computational performance is within 3% of the optimal value. We also extend our analysis to when workers' preference lists are "rotating" based on previous assignments, where worker will rank his or her last assignment as the least favorite job in the current period. We provide modified heuristics for solving the dynamic problem when $p = 3$, achieving 83% optimality.

At the end of the paper, we generalize the heuristics for other sets of preference lists. These preference lists are subsets of, or closely related to, the sufficient conditions for uniqueness of stable matching. For the static problem, our heuristics can provide practical instructions in assigning workers to jobs over multiple periods for a rich set of preference lists. Under the dynamic problem when our assumptions are applicable, our heuristics can also be implemented for providing practical guidelines to rotate workers through jobs over time.

We exploit the structure of heuristics and provide guidelines and for developing solution approaches under other settings. For workers at the top of jobs' preference lists, their stability requirements are more constraining. We generally satisfy stability related to these workers using as little utility as possible. Workers at the bottom of jobs' preference lists usually drive the objective function value, as we want to maximize the minimum individual utility. Fortunately, the stability requirements related to these workers are relatively easy to satisfy. We propose several approaches for balancing individual utilities with limited jobs of good quality.

Our research develops a framework to study the multi-period stable matching problem and considers solution approaches for varies practical settings. We provide analytical and computational results under these settings and provide insights and guidelines for practice. Our work potentially provide new directions for future research on multi-period stable matching.

CHAPTER 5
CONCLUSION

The thesis studies three practical problems that allocate resources to people with different kinds of preferences. We study each problem using mathematical models and focus on understanding the implications of people's preferences in decision making. Our research demonstrates how optimization models are used for incorporating various types of preferences and provides insights for designing practical and implementable tools for problems arise from complex systems.

In the first problem, we compare optimization methods to catchment methods in measuring the spatial access of patients, where the preferences of patients can be driven by distance, congestion (waiting time), and other factors such as income and insurance type. Specifically, we present decentralized optimization that assign patients to healthcare providers with an objective function to minimize total distance traveled and weighted congestion experienced by patients, using Nash equilibrium constraints to incorporate choices of patients. The catchment methods using analogy of gravity to model the attractiveness of healthcare providers to patients, and measure patient-to-provider ratios for each patient. We compare these two approaches analytically and using a case study to demonstrate the advantages of the optimization methods, which include

1. capturing the experience of patients instead of the opportunities,
2. having the flexibility to accommodate additional constraints, and
3. recognizing the system cascading effect under congestion.

The optimization models can be used to measure health access at different levels of details, and help decision makers to target areas with low access for potential intervention. To help researchers not familiar with the concept of optimization, we implement the decentralized optimization models using AMPL, which is an open access software language. In addition, we provide instructions and guidelines for submitting computer codes and utilizing the online NEOS Solvers for solving the optimization models, so that practitioners and analysts who are not familiar with the techniques can utilize the methods we have developed. We publish the research, which is available online, in a journal that is oriented towards the health services research community [1].

In the second problem, we demonstrate the need for information visibility in the public health supply chain when it comes to allocating flu vaccines in limited supply. Usually, vaccines are distributed to each location proportional to population, because public policies need to be transparent and fair. However, vaccine uptake rates are often different across geographical locations, due to differences in supply chain, socio-economic status, and personal beliefs.

We implement an agent-based simulation model (using C++) to capture the geographical aspects of the problem, especially the different uptake rates and commuting between population groups. The agent-based simulation model also allows us to capture the geographical spread of the disease and the variance in flu cases under different realizations of the pandemic. We compare strategies with (Population-and-inventory-based, or PIB) and without (Population-based, or PB) vaccine inventory information under the simulation framework with different parameter settings.

We find that under the PIB strategy with realistic parameters, there are fewer infections, more administered vaccine, less leftover vaccine, and higher service levels. States often have vaccine registry systems in placed for tracking immunization records, especially for children. The PIB strategy can potentially be implemented in the future, as the benefit of the PIB strategy outweighs the potential cost to track inventory levels at each location. Our geographically detailed simulation allows us to demonstrate the value of inventory information, showing state, local, and federal agencies the benefit of tracking location of vaccine administration. Moreover, our simulation can be expanded to incorporate other elements of vaccine allocation. For example, decision makers could use the inventory information from the registry to target educational efforts to increase uptake of vaccine, further reducing the number of infections and the amount of leftover vaccine. The research has been submitted for publication, with a focus on disseminating our results to practitioners.

In the third problem, we study the multi-period matching problem under complete preference lists, no ties, and relaxed stability. This problem addresses assignments of workers and jobs over time, which occurs regularly in organizations such as the World Food Program (WFP), the Navy, and in the National Medical Residency Program (NRMP). We address the case where preferences of workers stay the same over time, along with a model that incorporates preferences of workers rotating based on past assignments.

We provide an integer formulation with the objective function maximizing the minimum individual utility summed over periods and the multi-period stability constraint aggregating the stability constraints in each individual period. Our formulation studies the trade-offs between requiring a higher stability and a higher quality of assignments. We use a parameter α to indicate how stable a multi-period assignment is, which allows us to provide different assignments under applications with different requirement on stability or quality. Moreover, we provide solution approaches that are fast and with good computational performance under a special preference list, which represents a typical practical setting where assignments with good quality is hard to obtain.

We exploit the structure of these approaches and provide insights for generalizing our results to other cases, with the preference lists of the workers either staying the same or changing over time. Two organizations could potentially apply our work in their operations: the Navy and the World Food Program (WFP). For the Navy, each officer is ranked based on his or her performance on the current position and past experience, where the preference lists of each position can be arbitrary. We show that in this case we can apply our heuristics directly if officers submit complete preference lists and have no ties in their preference lists. For the World Food Program, we suggest staff on the top of jobs' preference lists are assigned enough preferred positions to satisfy stability. Then the WFP can negotiate with the staff members and let them take less preferred positions in other periods, leaving more preferred jobs to other workers to improve the quality of assignments. Our study potentially provide new directions for future research on multi-period stable matching, such as the study of incomplete preference lists, preferences with ties, etc.

In situations where centralized planners make decisions for the entire system, the solution can be very efficient. However, many real-life problems involve individual decision makers, and their preferences have to be taken into account. In this thesis we have focused on developing optimization and simulation models that incorporate preferences and decision making. At the same time, we consider how the results and solution approaches could be used to improve system efficiency and key objectives. This thesis has shown a few examples (in the areas of health access, vaccine allocation, and job assignments) on how to incorporate different preferences in consideration of system design, but there are still many avenues left to explore. We hope that this thesis can potentially provide new ideas and directions for future research in other areas, such as public health, supply chains, and operations management, where preferences are important in decision making.

Appendices

APPENDIX A
ACCESS

A.1 Parameter Selection and User Choice

In the optimization models, the congestion weight represents the trade-off between willingness to travel and willingness to wait, and it may be adapted to different applications. To identify a range of values that is reasonable for a given problem setting, we quantify several measures of model performance across the network. Below we describe such performance measures, provide the associated principles if applicable, and give calculation details for the CF case.

- Congestion Difference for Close Facilities: *Congestion at facilities that are very close to each other should be similar.* We quantify the absolute value of the difference of each pair of facilities within 50 miles of each other, and sum up the differences over the network.

- Distance Difference (or Congestion Difference) for Close Patients: *Cost experienced by patients who are very close to each other should be similar.* We calculate the variance in distance or congestion across individual visits originating in the same county and sum the values over the network.

- Variance in Distance (or Congestion) across Network: *Heterogeneous networks usually have some disparities in costs experienced; however, very extreme values may not be reasonable.* We quantify the mean distance traveled (or congestion experienced) for visits within a county, then we calculate the variance across the counties in the network.

- Distance Greater than Shortest Distance: *Distance traveled by patients should not* *be much greater than their shortest possible distances.* Distance to closest facility is compared to average distance traveled by each patient.

- Total Distance or Total Congestion: Calculated across a network by summing up the distance traveled or congestion experienced for each visit to a facility. These two measures are inversely related.

Figure A.1 shows the measures for the optimization models under different congestion weights, where the values are normalized [0,1] across results from both models. For patients whose visits are uncovered, we do not include those visits in the calculation of distance or congestion. Note that when the congestion factor is 0, the decentralized optimization is equivalent to the centralized optimization. In this case, the centralized optimization assignment reduces to finding the shortest distance between patients and hospitals. The far right corresponds to splitting congestion evenly among facilities. Thus, the total distance traveled increases with the congestion factor (although not by much), and the total congestion decreases significantly with the congestion factor.

The figure also shows that as the congestion weight increases, the variance of congestion across the network is decreasing, while the variance of distance across the network is increasing. For a very small congestion factor, distance is very important in the assignment to facilities, and thus facilities that are close to each other may have different levels of congestion. Using the principles above, there should be some differences in congestion and distance across the network, but not excessively large gaps, so we view congestion factors of around 10 as the most reasonable for this setting. The results for the centralized model with different congestion factors are also similar.

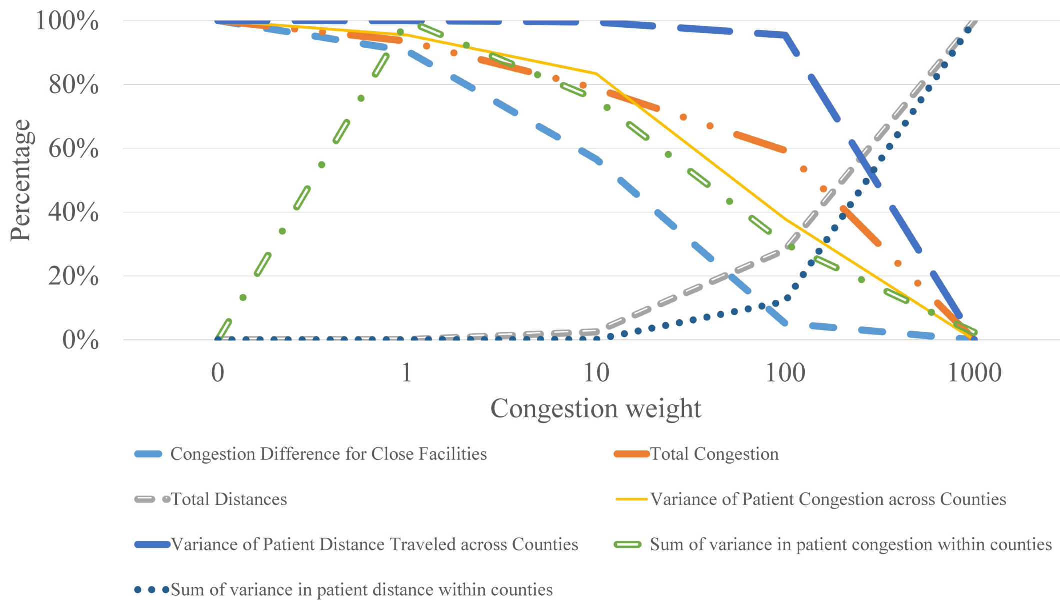

Figure A.1: Overall performance measures for different parameter settings of congestion for the decentralized optimization method for Cystic Fibrosis, with highlighted area of recommended values.

A.2 Other Variations on Optimization Model

Capacity: Some providers or facilities may have limited resources. This can be introduced by adding a capacity constraint to the basic model. Define c_j = capacity for provider j. The corresponding constraint is

$$\sum_{i=1}^{n} x_{ij} \leq c_j, \forall j \in [m]. \tag{A.1}$$

Unmet Demand: If resources in the network are limited, it may not be possible to meet all demand. In this case, the assignment constraint should be modified to $\leq$. In addition, to ensure that as much demand is met as possible, one can add constraints to ensure that for community i, the minimum service level requirement s_i is met, that is,

$$\sum_{j=1}^{m} x_{ij} \geq s_i v_i, \forall i = 1, \ldots, n. \tag{A.2}$$

Alternatively, one can add a penalty to the objective function for all visits not assigned.

Willingness to Travel: If patients are located too far from providers, they may not be as willing to travel to that provider. In the basic optimization model, the cost to travel is linear with distance. By adjusting the distance values, one can make the cost to travel nonlinear with distance, which represents a patient's higher willingness to travel to close distances. Particular adjustments can be chosen to match the weights of zones as used in the catchment models.

Patient or Provider Types: Some providers choose not to accept Medicaid patients (or limit how many they will accept), which can reduce the spatial access for those patients. One way to represent this in the model is by creating separate assignment variables for each patient type, and adding constraints to limit their assignment to providers with those preferences [92]. This allows the optimization approach to incorporate the link between affordability and spatial accessibility.

On the demand side, patients may have preferences for providers with certain characteristics, e.g., children and their caregivers may desire providers focused on pediatric care. One way to incorporate this is to adjust the travel cost to be relatively lower for providers of the preferred characteristics. This example shows how the optimization model can incorporate acceptability [7] in the measurement of access. A similar approach (adjusting distances) can be used to capture differences in patient mobility, e.g., for families with automotive vehicles or not.

Objective Function: In the Method section we describe a model with an objective function that has a particular congestion cost. Many other variations on congestion are possible, including linear with the number of visits at a facility, exponential with the number of visits, or others. More generally, many variations on the objective function are possible.

Interventions: Decision variables can be added to optimization models to represent whether or not a new facility should be located in a network at particular locations, whether or by how much to increase capacity, or other interventions. The interventions can be designed to optimize the overall system performance or to reduce the disparities among subpopulations.

A.3 Minimum cost network flow transformation for decentralized model

Decision Variables:

x_{ij} = the percentage of time that patients in location i visit facility j.

$y_{jk} = 1$ if the k^{th} visit is selected for facility j.

Parameters:

d_{ij} = distance between patient location i and facility j.

$w(d_{ij})$ = decay function value for distance d_{ij}.

v_i = demand of patient location i.

C_j = capacity at facility j.

$f_k = k$, the cost of marginal congestion for the k^{th} visit.

α = congestion weight.

Formulation 3 Decentralized optimization model

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{d_{ij}}{w(d_{ij})} x_{ij} v_i w(d_{ij}) + \alpha \sum_{j=1}^{m} \frac{1}{C_j} \sum_{k=1}^{n} f_k y_{jk} \tag{A.3}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} x_{ij} v_i = v_i, \forall i \in [n] \tag{A.4}$$

$$\sum_{i=1}^{n} x_{ij} v_i w(d_{ij}) = \sum_{k=1}^{n} y_{ik}, \forall j \in [m] \tag{A.5}$$

$$x_{ij} \geq 0, \forall i \in [n] \text{ and } \forall j \in [m] \tag{A.6}$$

$$y_{ik} \in [0,1], \forall i \in [n] \text{ and } \forall j \in [m] \tag{A.7}$$

A.4 Analytical Results: networks with overlapping areas

Result 4: Optimization models show higher accessibility in non-overlapping areas. It can be difficult to understand model differences across complex networks like we study in section 2.3.2. Thus we analyze one more simulated system (shown in Figure A.2) that can assist in making comparisons between the 2SFCA approaches and optimization models.

1. When the population density is homogeneous over the network:

   Consider System I with two facilities, each with a population surrounding them in a circle of radius R. The distance between the two facilities is also R, so some population resides between both facilities. Define the decay function

   $$w(d_{ij}) = e^{-d_{ij}}, 0 \leq d_{ij} \leq R. \tag{A.8}$$

   The density of the areas is 1 unit per square mile and the supply C is the same in each facility. We will compare composite measures across the network in Figure A.2.

   For 2FSCA the physician-to-population ratio at each facility is $S = \frac{C}{V}$, where $V = \int_0^{2\pi} \int_0^r e^{-r} r dr d\theta$ denotes the number of visits. For the population inside the catchment of only one facility, each patient's accessibility can be calculated by

   $$S_{P_s}^r = e^{-r} \frac{C}{V}, \tag{A.9}$$

   where r is the distance between the patient and the facility. For the population in overlapping catchment areas, a patient's accessibility can be calculated as

   $$S_{P_o}^{r_1, r_2} = (e^{-r_1} + e^{-r_2}) \frac{C}{V}, \tag{A.10}$$

   where r_1 is the distance to the first facility and r_2 is the distance to the second facility. We also have

   $$r_1 + r_2 \leq R, \text{ and } r_1, r_2 \leq R, \tag{A.11}$$

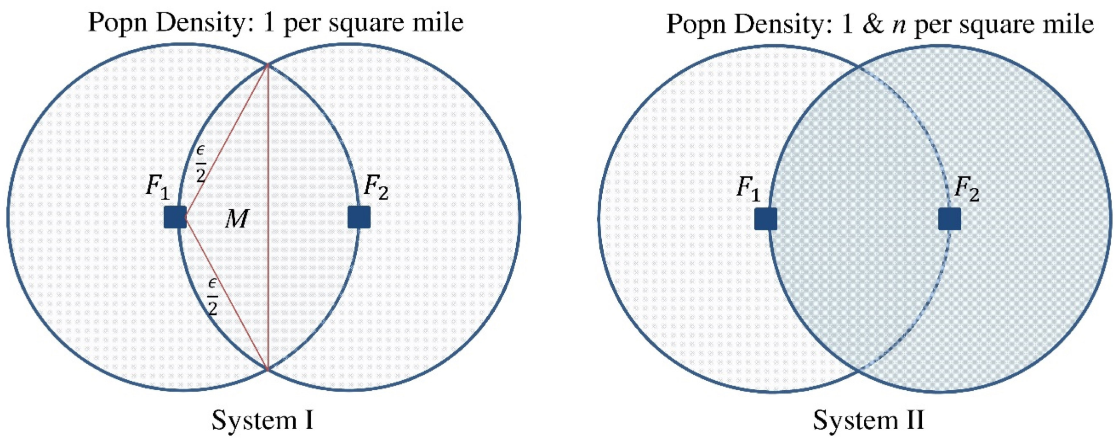

Figure A.2: Systems I and II have populations distributed in circles around facility 1 and 2. In I, the density is 1 person per square mile, and in II the density is 1 and n per square mile for the left and right circles, respectively. In I, the figure indicates the locations of populations M and ϵ used in the calculations.

which implies that

$$2e^{-R} \leq e^{-r_1} + e^{-r_2} \leq 2e^{-\frac{R}{2}}. \tag{A.12}$$

For the optimization models, we initially use a congestion weight such that patients will visit their closest facility. The congestion at each facility is

$$F = \frac{\frac{2}{3}V + M}{C}, \tag{A.13}$$

where

$$M = \int_0^{1/3\pi} -\frac{R}{2\cos\theta} e^{-R/2\cos\theta} d\theta < \frac{V}{3}. \tag{A.14}$$

For the population inside the catchment of only one facility, the patients congestion is $F_{P_s} = F$. For the population in the overlapping catchment areas the congestion experienced by each patient is $F_{P_o} = F$.

If a patient is inside a single circle, then the optimization model shows higher accessibility than the 2SFCA approaches since $F_{P_s} = F < 1/S$. This is true for larger congestion weights if there is no decay function or if the congestion weights are extreme. The result occurs because visits are over-counted in the 2SFCA methods, while the optimization model is capturing the cost associated with user experience. For patients in the overlapping areas, we find that which method estimates higher accessibility depends on the value of radius R. If $R < ln(4/3)$, then the accessibility for patients in the middle is:

$$S^r_{P_o} \geq \frac{2Ce^{-R}}{V} > \frac{3C}{2V} > \frac{1}{F_{P_o}}, \tag{A.15}$$

which implies that the overall range of accessibility in the optimization model is smaller than the 2SFCA methods, so the access appears smoother. The R values that are small represent dense areas.

2. When the population density is non-homogeneous over the network, we consider system II. The E2SFCA facility accessibility measures are

$$S_1 = \frac{C}{V + (a-1)(V/3 + \epsilon_1)} \tag{A.16}$$

$$S_2 = \frac{C}{aV} \tag{A.17}$$

$$\epsilon_1 = \int_{\pi/3}^{\pi/2} \int_0^{2R\cos\theta} e^{-r} r dr d\theta \tag{A.18}$$

For a patient inside the catchment of facility 1 and 2 only, the patient level measures are:

$$S^r_{P_1} = e^{-r} S_1, \tag{A.19}$$

$$S^r_{P_2} = e^{-r} S_2. \tag{A.20}$$

For a patient inside the overlapping area:

$$S_{P_o}^{r_1,r_2} = e^{-r_1}S_1 + e^{-r_2}S_2 \tag{A.21}$$

$$r_1 + r_2 \leq R \tag{A.22}$$

$$r_1, r_2 \leq R, \tag{A.23}$$

which implies that

$$2e^{-R} \leq e^{-r_1} + e^{-r_2} \leq 2e^{-R/2}. \tag{A.24}$$

The facility level congestion measures for Shortest Distance are:

$$F_1 = \frac{\frac{2V}{3} + aM + (a-1)\epsilon}{C}, \tag{A.25}$$

$$F_2 = \frac{a\left(\frac{2V}{3} + M\right)}{C}. \tag{A.26}$$

For a patient inside the catchment of facility 1 and 2 only: $F_{P_1} = F_1$ and $F_{P_2} = F_2$. For a patient inside the overlapping area: $F_1 \leq F_{P_o} \leq F_2$. For this system, again we have at the facility level,

$$F_1 < \frac{1}{S_1} \text{ and } F_2 < \frac{1}{S_2}. \tag{A.27}$$

At the patient level, it is obvious that the following conditions hold:

$$S_{P_1}^r < F_{P_1} \text{ and } S_{P_2}^r < F_{P_2}. \tag{A.28}$$

If $R < \ln(2/3 + (2M + \epsilon)/V)$, then the access under Shortest Distance has a smaller range (i.e., "smoother"). This R is guaranteed to exist, since $\epsilon' < \epsilon$ and $M' < M$

implies

$$\frac{V}{3} < 2M + \epsilon \implies \frac{2}{3} + \frac{2M + \epsilon}{V} > 1. \tag{A.29}$$

A.5 Incidence matrix for race/ethnicity

Table A.1: Incidence rate of Cystic Fibrosis by race/ethnicity [93, 94].

Race/Ethnicity	Incidence
White(non-Hispanic)	1/3,000
Hispanic White	1/13,5000
African American	1/15,000
Asian	1/30,000

A.6 Additional Figures for Case Study

Figure A.3 shows the simulated CF population and the CF centers in the continental United States. Figure A.4 shows the histograms of optimization model results.

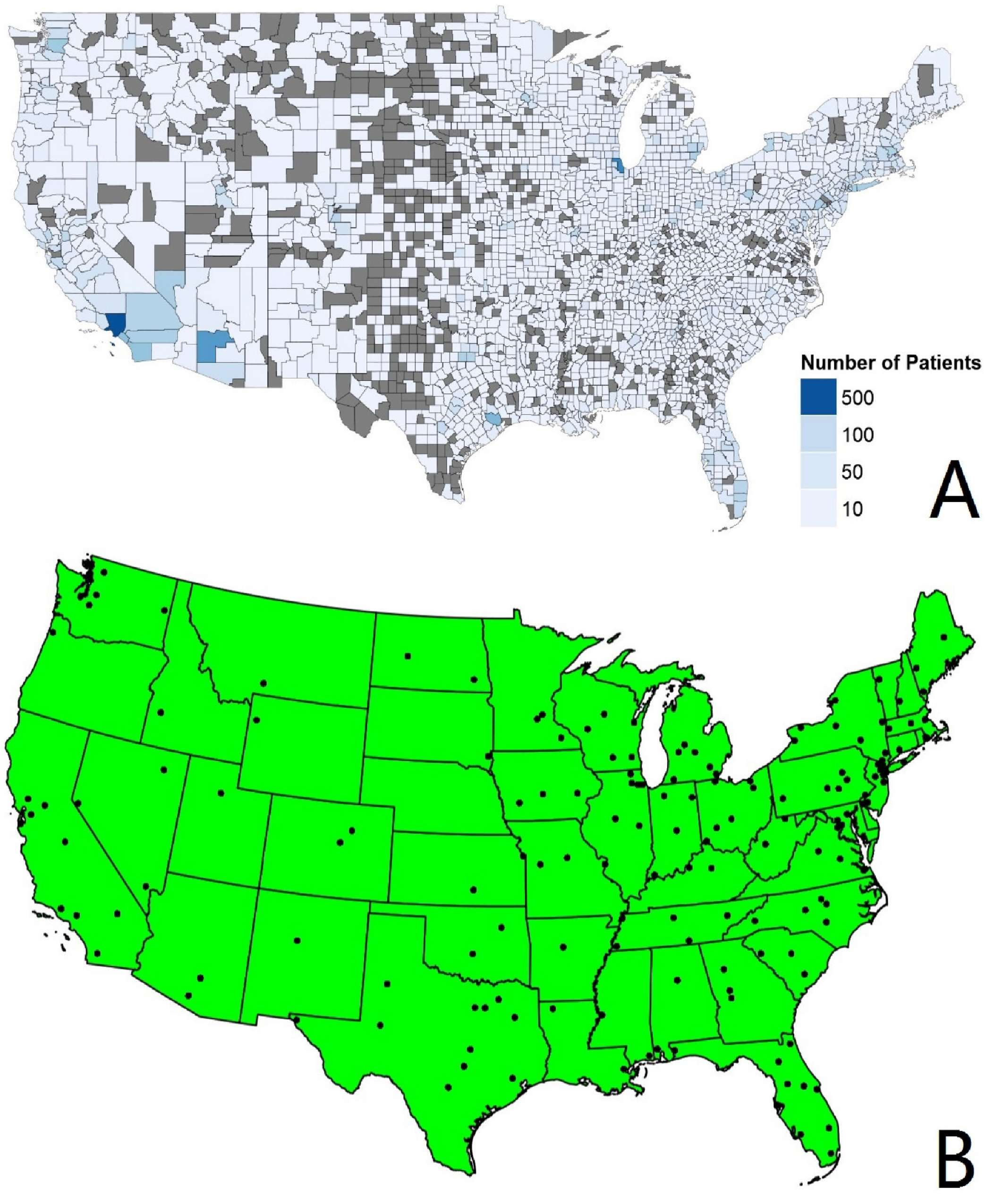

Figure A.3: Distribution of simulated population of CF patients in US where uncolored counties have no patients (A); and locations of CF centers (B).

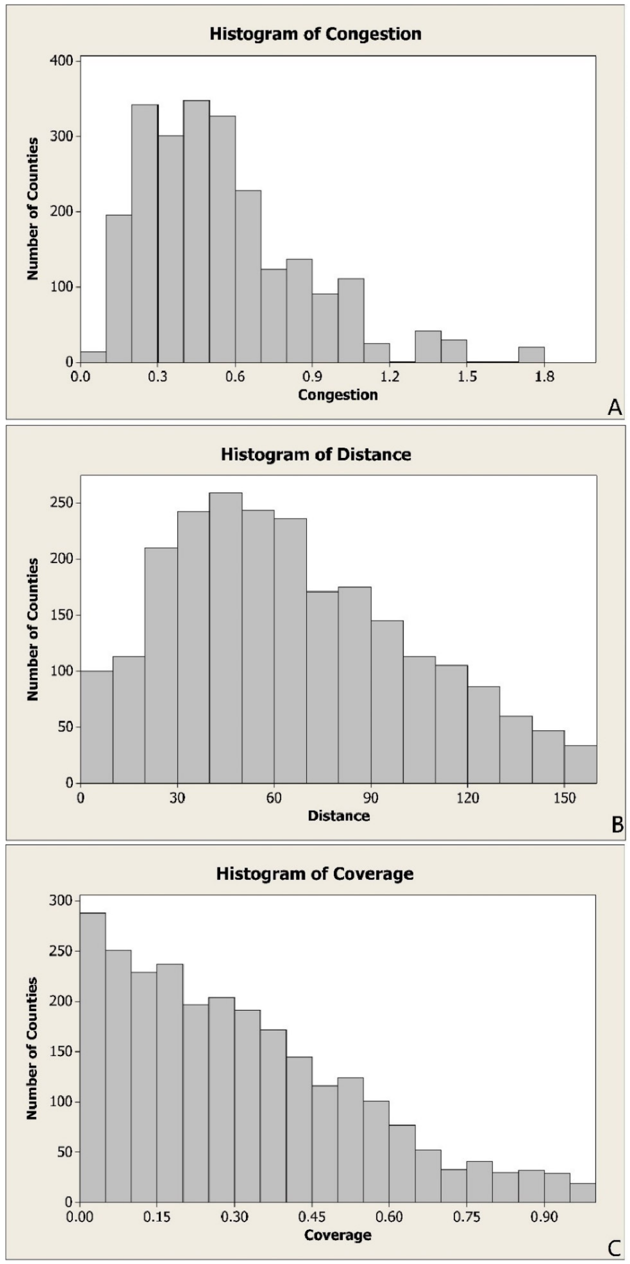

Figure A.4: Histograms of optimization model results. (A) Congestion, (B) distance, and (C) coverage.

A.7 Model specifics: Cystic Fibrosis Implementation

Figure A.5 displays the percentage visits to Cystic Fibrosis care centers from 1997 to 2013 of different distances along with the visits quantified by an exponential decay function with parameter = 0.02.

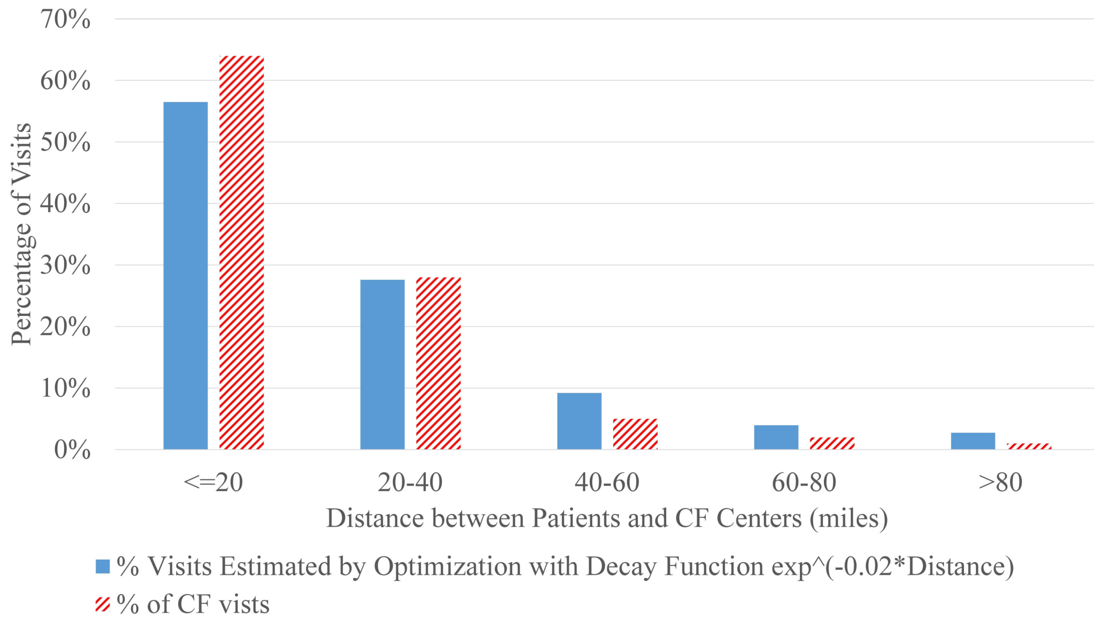

Figure A.5: Actual CF visits compared to visits from optimization

We estimate the number of visits for each patient-CF center pair i,j based on an exponentially decaying function $v_{ij} = 10e^{-0.02d_{ij}}, d_{ij} \leq 150$ miles, and $v_{ij} = 0, d_{ij} > 150$ miles. For all patients with family income below two times the Federal Poverty Line (FPL), $v_{ij} = 0$ for all centers located in other states. All parameters are summarized in Table A.2.

For the 2SFCA methods, the three catchment zones are defined by 0-50, 50-100, and 100-150 miles. We set $v_i = 10, c_j = 1,500$ for $j \in [m]$, and we quantify w_i on zone $i = 1, 2, 3$ as

$$w_1 = e^{-0.02*(0+50)/2} = 0.607,$$

$$w_2 = e^{-0.02*(50+100)/2} = 0.223, \text{ and}$$

$$w_3 = e^{-0.02*(100+150)/2} = 0.082.$$

Table A.2: Cystic Fibrosis implementation parameters.

	2SFCA Methods	Optimization Model
Zone 1	$w_1 = 0.607$	$d_{ij}^{adj} = d_{ij}e^{0.02d_{ij}}, \forall d_{ij} \leq 150$ miles
Zone 2	$w_2 = 0.223$	and
Zone 3	$w_3 = 0.082$	$d_{ij}^{adj} = 9,999, \forall d_{ij} > 150$ miles
Number of counties	2,568	
Number of facilities	208	208 + dummy location
Capacity/facility	1,500	

APPENDIX B

VACCINE

B.1 Illustration of disease modeling

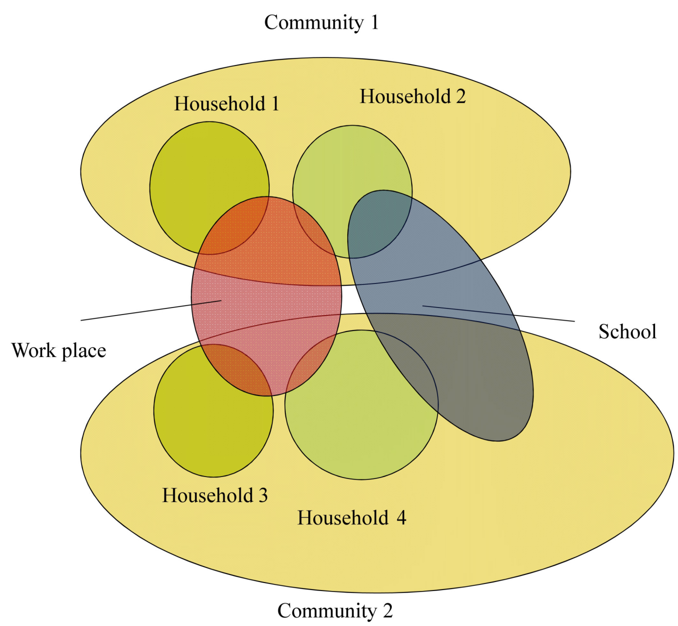

Figure B.1: Example of a contact network.

$p_A = 0.4$ for adults (19-64) and 0.25 for others [95, 96, 97, 98];

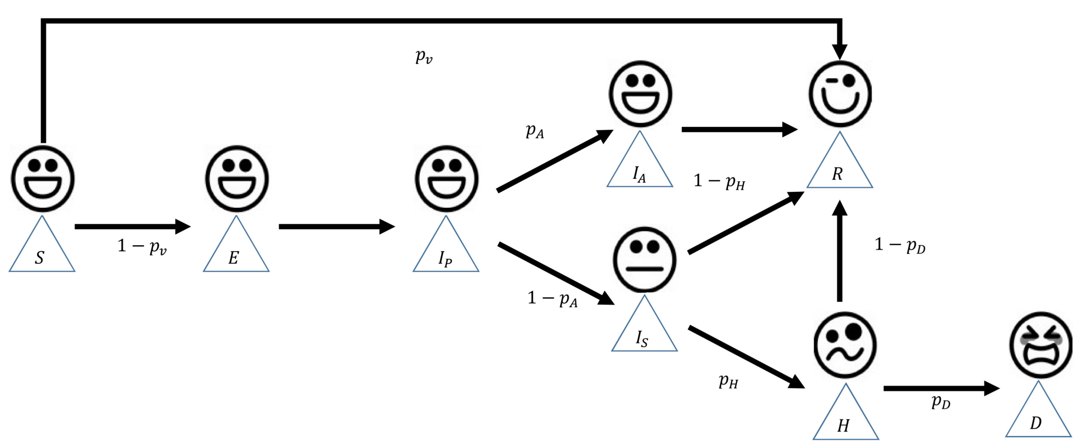

Figure B.2: Natural history of the disease.

$p_H = 0.18$ for children ≤ 5 years; 0.12 for elderly (65+) and 0.06 for others [97, 98];

$p_D = 0.344$ for elderly and children ≤ 5 years and 0.172 for others [98, 99];

$p_v = 0.75$ for individuals who have been vaccinated two weeks ago, 0 for others [38];

Duration of $E + I_P \sim Weibull(1.48, 0.47)$, including an offset of 0.5 days [98, 100];

Duration of $I_P = 0.5$ days [98, 100];

Duration of $I_S \sim Exponential(2.73)$ [98];

Duration of $I_A \sim Exponential(1.64)$ [98];

Duration of $I_H \sim Exponential(14)$ [98, 100];

B.2 Detailed description of allocation policies

We assume that the vaccine supply is available at the beginning of a given week w and the total amount of vaccine available is T_w. For each census tract i in week w, we have the population p_i, vaccine inventory level $v_i^w(v\_i^1 = 0, \forall i)$ and the total amount of vaccine shipped prior to week t_i^w. We calculate the size of potential susceptible population for each census tract as $S_i^w = p_i - \sum_{k<w} t_i^k$. Let r_i^w denote the vaccine requirement in census tract i in week w. Under the PB strategy $r_i^w = p_i$, for each census tract i and each week w.Under the PIB strategy, $r_i^1 = p_i$ for each census tract i. For $w \geq 2$, $r_i^w = \begin{cases} t_i^w, & v_i^w \leq 0 \\ 0, & v_i^w > 0 \end{cases}$. Note that the requirement under PIB is in line with the demand, i.e., the number of people who are willing to receive the vaccine, who have not been vaccinated, and who have not been infected and recovered. Both policies allocate $a_i^w = \min \frac{r_i^w}{\sum_i r_i^w} \cdot T_w, S_i^w$ amount of vaccine to each census tract i. In both strategies, fractional levels of vaccine are rounded down to the nearest integer and if there is any vaccine left, they are distributed proportionally to census tracts that still demand vaccine.

B.3 Parameters of the simulation model

During the 2009-2010 influenza vaccine campaign, the vaccine supply (about 120 million doses) was sufficient to cover about 40% of the US population (about 300 million). Vaccine was delivered starting around week 40 of 2009 [101]. For the seasonal influenza vaccine, the uptake rate during the 2009-2010 season was 40.4% for adults, and 43.4% for children [102]. The solid line in Figure B.3 shows the percentage of total hospital visits that are due

Table B.1: Parameters for vaccination.

Start week	4 and 7
Total supply	20%, 40%, 60%, and 80% of total population
Uptake rate	1) half 25% half 75%; 2) half 0% half 100%; 3) Uniform$(0\%, 100\%)$
Distribution horizon	4, 8, and 12 weeks
Effectiveness	75%
R_0	1.5, 1.8, and 2.0

to influenza like illnesses (ILI) during the 2009-2010 influenza pandemic and the two double lines indicate the time periods for vaccination in our simulation when the vaccination start week is four or seven and vaccines are distributed for eight weeks. As a comparison, the dashed straight line is the national baseline for the percentage of hospital visits that are due to ILI during the regular influenza season. Week one in Figure B.3 corresponds to the 35th week of 2009.

B.4 Analysis on number of iterations

In this study, the number of iterations for each parameter setting is $N = 50$. Although the same ten uptake rate instances are used for all the five networks, the 50 IARs pass all normality test and there are no correlations between networks with the same uptake rate instance. The maximum sampled standard deviation for all scenarios is $\sigma = 0.015$.

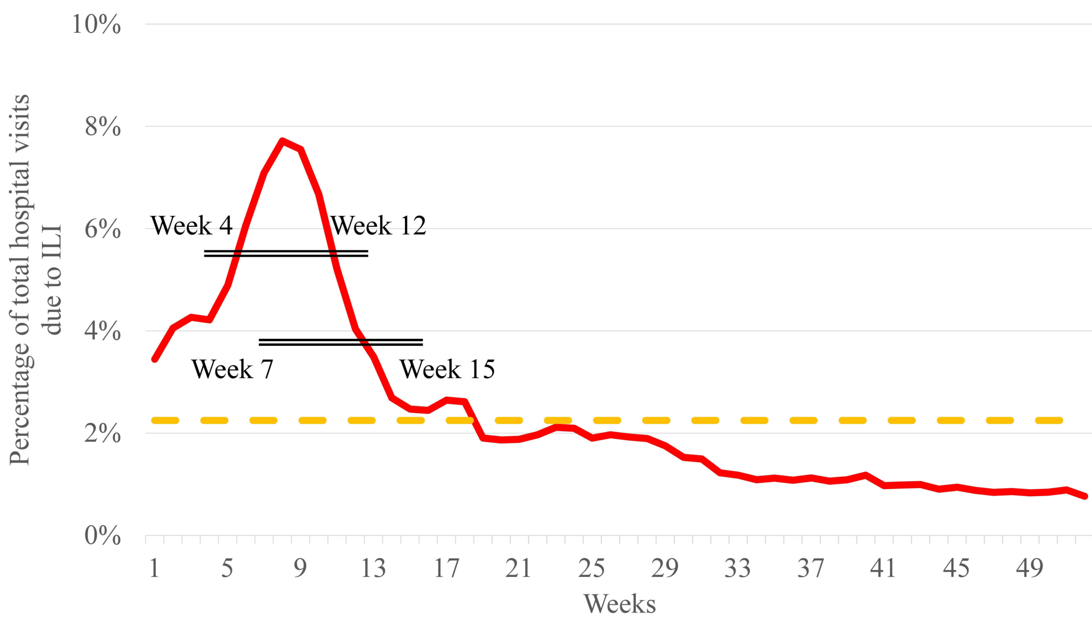

Figure B.3: Percentage of visits for influenza like illness 2009 to 2010 season [103].

On a confidence interval of 95%, which corresponds to a z-value of 1.96, the width is $z \cdot \frac{\sigma}{\sqrt{N}} = 0.004$ or 0.4% of total population.

B.5 Detailed comparison between strategies under multiple scenarios

For each combination of vaccine supply duration in weeks, vaccination start week and total vaccine supply, we measure the IAR, the amount of vaccine inventory at the end of the epidemic and the amount of vaccine administrated for PB and PIB in Table B.2 to B.10. All p-values and the corresponding confidence intervals are under two-sample t-test with confidence level 0.95.

B.6 No vaccination vs. vaccination (vaccine supply is 40% population)

B.6.1 IAR when no vaccine is available

Figure B.4 shows that without vaccine, influenza virus spreads to every county of Georgia. Counties closer to the City of Atlanta tend to have a higher IAR.

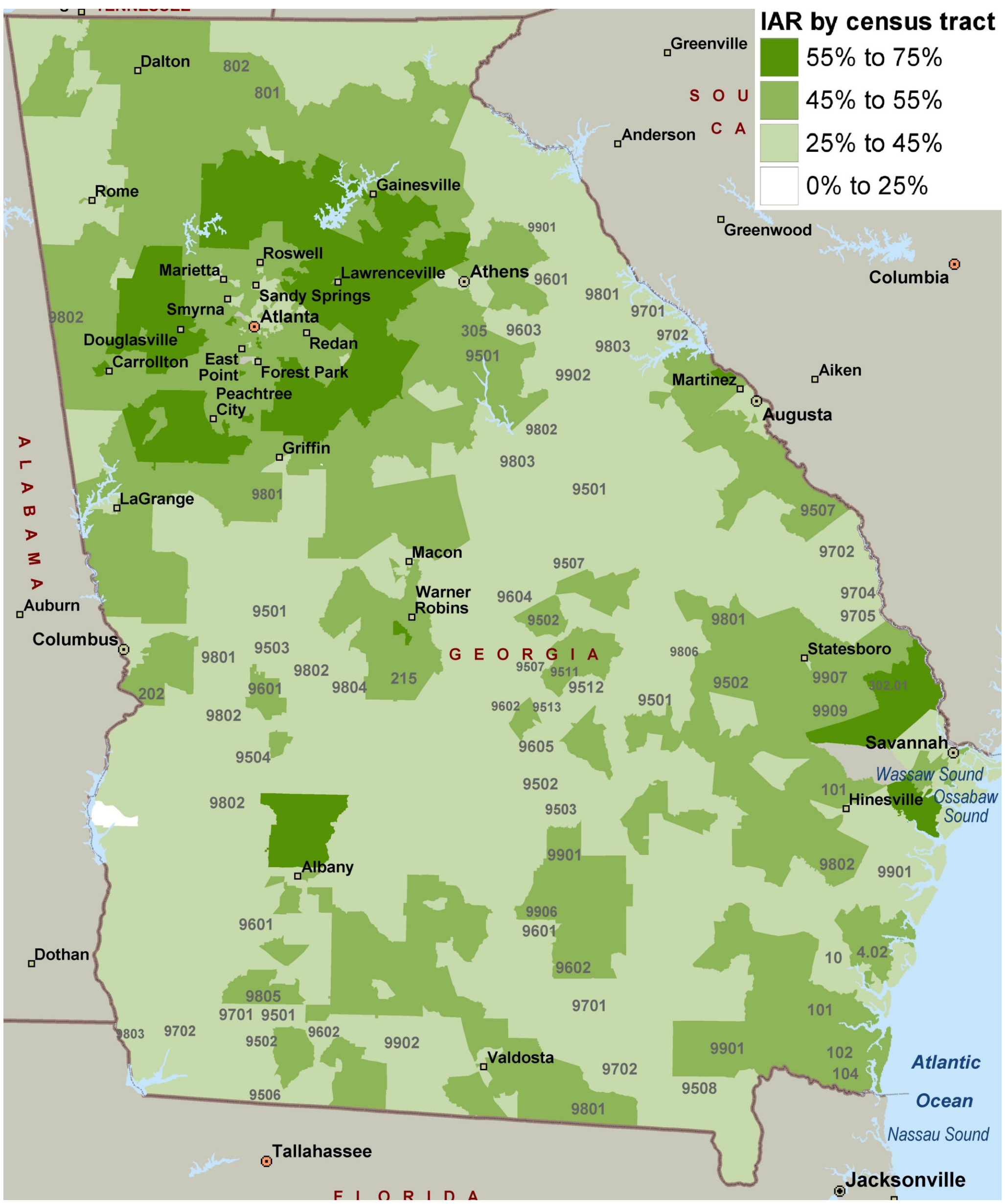

Figure B.4: Total attack rate of each census tract of state of Georgia under no vaccination.

B.6.2 Peak prevalence and IAR

Prior studies suggest that the overall impact of vaccination can be sensitive to the start time of vaccination and the amount of vaccine available [104].

Figure B.5 shows the prevalence (percentage of population infected) under no vaccination and vaccination following the PB strategy when the total vaccine supply is equal to 40% of the total population.

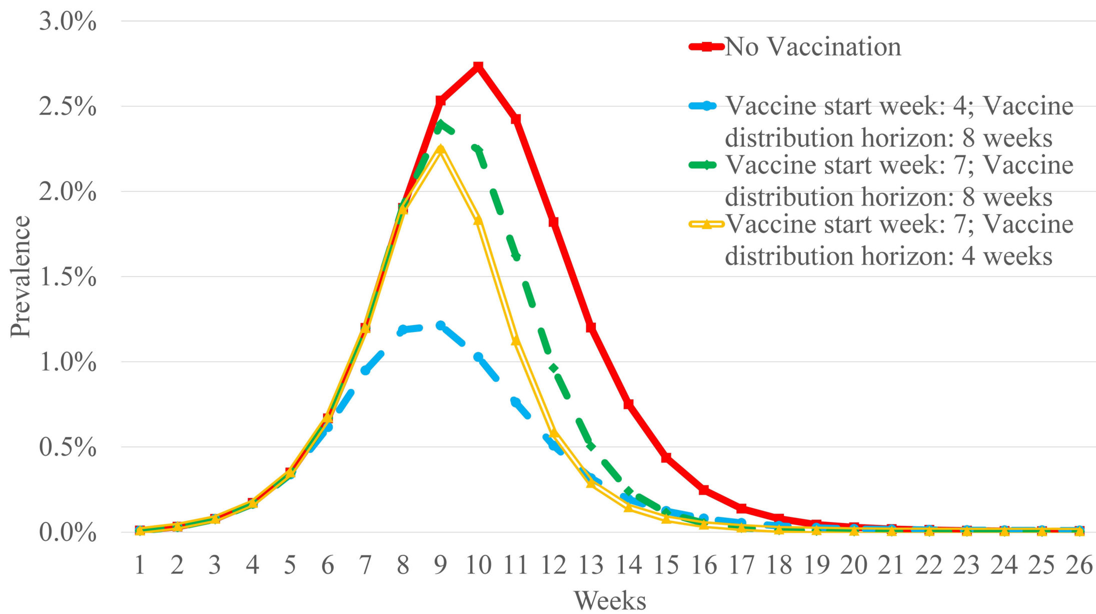

Figure B.5: Disease prevalence in Georgia under (i) no vaccination and (ii) PB strategy when the vaccination start week is week four or seven, and the vaccine distribution horizon is four weeks or eight weeks.

B.6.3 IAR Comparison

Figure B.6 compares the IAR under the PB strategy for different vaccine supply levels. Figure B.6A shows the comparison of IAR when the vaccination start week is four or seven, and the vaccine allocation horizon is four weeks; Figure B.6B is similar but the vaccine delivery horizon is eight weeks. The IAR is lower for each scenario in Figure B.6A compared to B.6B. Starting vaccination in week four instead of week seven reduces the IAR from 37.1% to 23.4% under PB when the vaccine supply is sufficient to cover 40%

of population and the vaccine distribution horizon is eight weeks.

The importance of the vaccination start week can even outweigh the importance of the total vaccine supply. For example, consider the two scenarios where (i) vaccine supply is equivalent to 20% of the population, vaccine distribution horizon is four weeks, and vaccine start week is week four (IAR = 27.8% for PB), and (ii) vaccine supply is 80% of the population, vaccine distribution horizon is eight weeks, and vaccine start week is week seven (IAR = 30.8% for PB). Even though the vaccine inventory is four times higher under (ii) versus (i), IAR is higher under (ii) because most of the vaccine arrives after the peak. These observations about the importance of the vaccination start week are consistent with the previous findings in the literature. [24, 105, 106].

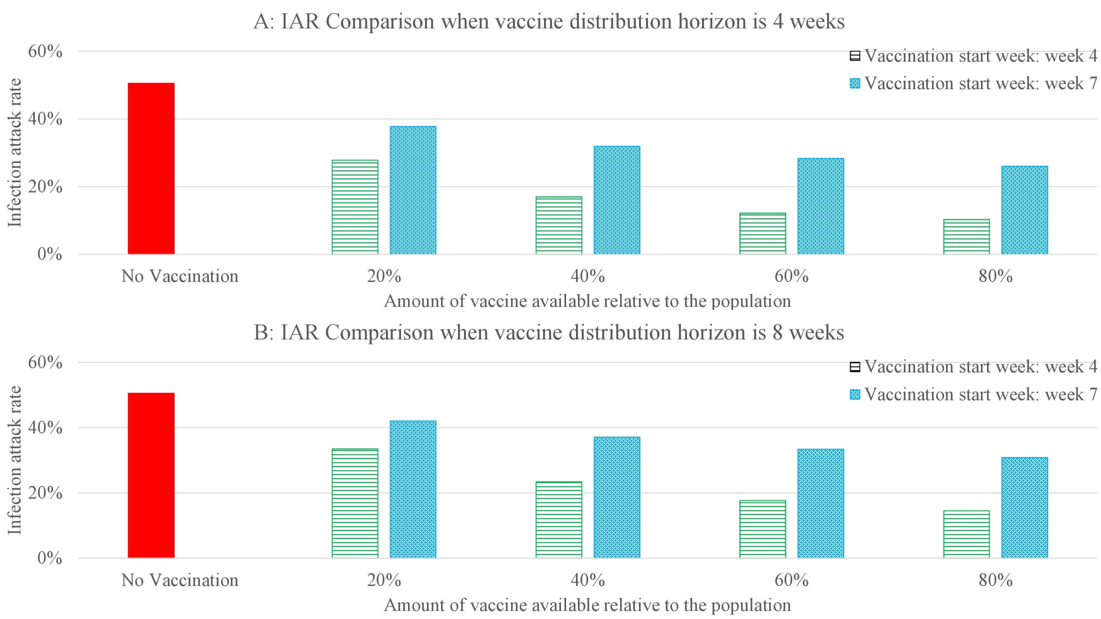

Figure B.6: IAR comparison under different vaccine supply levels when the vaccination start week is four or seven. B.6A: vaccine is distributed over four weeks. B.6B: vaccine is distributed over eight weeks.

B.7 Additional analysis and discussion on PB vs. PIB

B.7.1 Service level

For a given census tract, we defined the service level as the total number of vaccine administered divided by the population willing to receive the vaccine. We perform t-tests to compare the average service levels for PIB and PB. At a 0.05 significance level, for census tracts with 25% uptake rate, the 95% confidence interval on PIB-PB is 0.0% to 0.1% with $p = 0.0058$; for census tracts with 75% uptake rate, the 95% confidence interval on PIB-PB is 17.7% to 17.8% with $p < 0.0001$.

B.7.2 Network effect

Figure B.7 shows the network effects when the uptake rate is correlated with education level (census tracts with 82.4% population (age ≥ 18) with high school or higher degrees have uptake 75%, others have 25%). The IAR under PIB is 0.2% (95% CI, 0.1% to 0.3%, $p = 0.0017$) less than that under PB on average for census tracts with 25% uptake rate and 0.9% (0.8% to 1.0%, $p < 0.0001$) for census tracts with 75% uptake rate.

B.7.3 Herd Immunity

Interestingly, when vaccine supply is low (20% of population, distributed in week four at once in Figure B.6), the reduction in IAR is equivalent to 25% of the population (from 50% to 25%), which is more than the vaccine supplied. However, when vaccine supply is high (80% supply but same otherwise), the effectiveness of an individual vaccine is less (IAR reduces from 50.5% down to 8.0%). This is at least in part because some of the vaccines are unused since the average uptake rate is 50%.

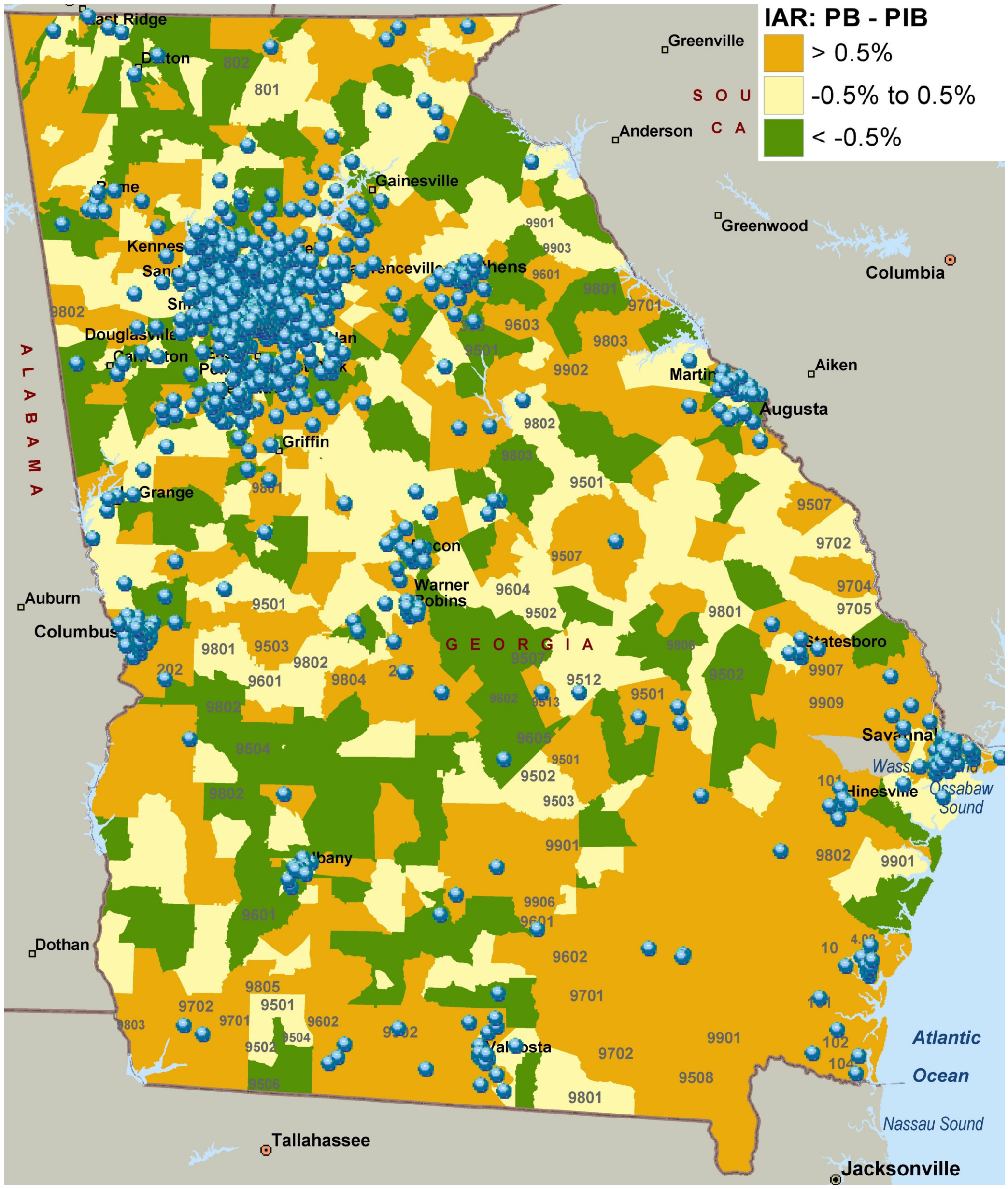

Figure B.7: IAR difference between PIB and PB strategies at county level. PIB has higher IAR in darker counties and lower IAR in lighter counties. Dots are census tracts with 75% uptake rate.

Table B.2: Comparison on total attack rate under UTR_1.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	27.8%	27.8%	0.1%	-0.2%	0.3%	0.6121
		40%	17.0%	15.4%	1.6%	1.3%	1.9%	0.0000
		60%	12.1%	11.0%	1.1%	0.8%	1.5%	0.0000
		80%	10.3%	10.1%	0.2%	-0.1%	0.4%	0.2978
	7	20%	37.8%	37.9%	-0.1%	-0.5%	0.2%	0.4145
		40%	31.9%	31.2%	0.7%	0.3%	1.2%	0.0026
		60%	28.3%	27.9%	0.4%	-0.2%	1.0%	0.1826
		80%	26.0%	26.2%	-0.2%	-0.8%	0.4%	0.5514
8	4	20%	33.5%	33.5%	-0.1%	-0.4%	0.2%	0.7026
		40%	23.4%	22.4%	1.0%	0.6%	1.4%	0.0000
		60%	17.7%	16.5%	1.2%	0.8%	1.6%	0.0000
		80%	14.5%	13.9%	0.7%	0.2%	1.1%	0.0024
	7	20%	42.1%	42.1%	-0.1%	-0.3%	0.2%	0.6651
		40%	37.1%	36.7%	0.3%	0.0%	0.7%	0.0804
		60%	33.4%	33.0%	0.4%	-0.1%	0.9%	0.1348
		80%	30.8%	30.9%	-0.1%	-0.6%	0.4%	0.6408
12	4	20%	37.8%	37.9%	0.0%	-0.2%	0.2%	0.9822
		40%	28.8%	28.4%	0.4%	0.0%	0.8%	0.0540
		60%	22.5%	22.1%	0.4%	0.0%	0.8%	0.0663
		80%	18.7%	18.0%	0.7%	0.2%	1.2%	0.0030
	7	20%	44.1%	44.0%	0.1%	-0.1%	0.3%	0.3899
		40%	39.8%	39.8%	0.0%	-0.2%	0.3%	0.7486
		60%	37.0%	36.8%	0.1%	-0.3%	0.6%	0.4969
		80%	34.4%	34.3%	0.1%	-0.4%	0.6%	0.6855

Table B.3: Comparison on total attack rate under UTR_2.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	39.4%	33.4%	6.0%	5.8%	6.2%	0.0000
		40%	31.4%	24.7%	6.7%	6.4%	6.9%	0.0000
		60%	26.1%	21.4%	4.7%	4.4%	5.0%	0.0000
		80%	22.8%	20.5%	2.2%	1.9%	2.6%	0.0000
	7	20%	43.8%	40.9%	2.9%	2.7%	3.2%	0.0000
		40%	39.5%	36.3%	3.1%	2.7%	3.6%	0.0000
		60%	36.4%	33.6%	2.7%	2.2%	3.2%	0.0000
		80%	34.2%	32.6%	1.6%	1.1%	2.0%	0.0000
8	4	20%	41.9%	36.8%	5.1%	4.9%	5.3%	0.0000
		40%	35.6%	29.2%	6.4%	6.0%	6.7%	0.0000
		60%	30.9%	24.9%	5.9%	5.5%	6.3%	0.0000
		80%	27.2%	23.1%	4.0%	3.7%	4.4%	0.0000
	7	20%	45.9%	43.9%	2.1%	1.9%	2.3%	0.0000
		40%	43.0%	40.0%	3.0%	2.7%	3.2%	0.0000
		60%	40.4%	37.7%	2.7%	2.4%	3.0%	0.0000
		80%	38.5%	35.8%	2.8%	2.3%	3.2%	0.0000
12	4	20%	44.0%	39.9%	4.1%	3.9%	4.3%	0.0000
		40%	39.2%	33.3%	5.8%	5.6%	6.1%	0.0000
		60%	34.9%	28.8%	6.0%	5.7%	6.4%	0.0000
		80%	31.6%	26.1%	5.6%	5.2%	5.9%	0.0000
	7	20%	47.0%	45.3%	1.7%	1.6%	1.9%	0.0000
		40%	44.7%	42.1%	2.6%	2.4%	2.8%	0.0000
		60%	42.6%	40.1%	2.5%	2.2%	2.8%	0.0000
		80%	40.8%	38.1%	2.7%	2.4%	3.0%	0.0000

Table B.4: Comparison on total attack rate under UTR_3.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	29.9%	28.7%	1.2%	1.0%	1.4%	0.0000
		40%	18.5%	16.5%	2.0%	1.6%	2.3%	0.0000
		60%	13.8%	12.7%	1.1%	0.8%	1.4%	0.0000
		80%	11.9%	11.5%	0.4%	0.1%	0.8%	0.0230
	7	20%	38.8%	38.3%	0.4%	0.1%	0.8%	0.0223
		40%	32.5%	31.7%	0.8%	0.2%	1.3%	0.0054
		60%	29.5%	28.4%	1.1%	0.6%	1.6%	0.0001
		80%	27.5%	26.7%	0.7%	0.2%	1.3%	0.0098
8	4	20%	34.8%	33.9%	0.9%	0.6%	1.2%	0.0000
		40%	25.0%	23.4%	1.6%	1.2%	2.0%	0.0000
		60%	19.2%	17.5%	1.6%	1.2%	2.0%	0.0000
		80%	16.2%	15.1%	1.1%	0.7%	1.4%	0.0000
	7	20%	42.5%	42.3%	0.2%	-0.1%	0.4%	0.1828
		40%	37.9%	37.1%	0.8%	0.5%	1.2%	0.0000
		60%	34.6%	33.6%	1.0%	0.6%	1.5%	0.0000
		80%	32.0%	31.7%	0.3%	-0.2%	0.8%	0.2278
12	4	20%	38.7%	38.1%	0.6%	0.4%	0.8%	0.0000
		40%	30.4%	29.0%	1.4%	1.0%	1.8%	0.0000
		60%	24.7%	22.8%	1.9%	1.4%	2.3%	0.0000
		80%	20.5%	18.8%	1.7%	1.2%	2.3%	0.0000
	7	20%	44.4%	44.4%	0.1%	-0.1%	0.3%	0.5262
		40%	40.4%	40.0%	0.4%	0.1%	0.6%	0.0171
		60%	37.8%	37.0%	0.8%	0.4%	1.2%	0.0001
		80%	35.0%	34.4%	0.6%	0.1%	1.0%	0.0126

Table B.5: Comparison on leftover inventory under UTR_1.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0006
		40%	7.9%	2.9%	5.0%	4.9%	5.0%	0.0000
		60%	17.7%	11.5%	6.3%	6.1%	6.4%	0.0000
		80%	31.5%	11.3%	20.2%	20.1%	20.4%	0.0000
	7	20%	0.1%	0.1%	0.0%	0.0%	0.0%	0.9814
		40%	8.5%	3.4%	5.1%	5.0%	5.2%	0.0000
		60%	18.5%	15.1%	3.4%	3.2%	3.6%	0.0000
		80%	34.6%	14.3%	20.3%	20.1%	20.5%	0.0000
8	4	20%	0.1%	0.1%	0.0%	0.0%	0.0%	0.0000
		40%	8.3%	1.5%	6.7%	6.7%	6.8%	0.0000
		60%	18.0%	6.0%	12.1%	11.9%	12.2%	0.0000
		80%	33.1%	11.8%	21.3%	21.1%	21.6%	0.0000
	7	20%	1.0%	0.7%	0.3%	0.3%	0.4%	0.0000
		40%	9.4%	3.3%	6.1%	6.0%	6.3%	0.0000
		60%	20.8%	8.9%	11.9%	11.7%	12.2%	0.0000
		80%	38.4%	8.3%	30.0%	29.8%	30.2%	0.0000
12	4	20%	0.7%	0.4%	0.3%	0.2%	0.3%	0.0000
		40%	8.8%	1.5%	7.3%	7.2%	7.4%	0.0000
		60%	18.7%	5.9%	12.8%	12.6%	13.0%	0.0000
		80%	35.2%	7.3%	27.9%	27.7%	28.1%	0.0000
	7	20%	1.9%	0.6%	1.3%	1.3%	1.3%	0.0000
		40%	10.4%	5.4%	4.9%	4.8%	5.1%	0.0000
		60%	23.8%	7.4%	16.4%	16.2%	16.6%	0.0000
		80%	41.5%	7.3%	34.2%	34.0%	34.4%	0.0000

Table B.6: Comparison on leftover inventory under UTR_2.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	10.2%	2.5%	7.6%	7.5%	7.7%	0.0000
		40%	20.1%	5.0%	15.1%	14.9%	15.3%	0.0000
		60%	30.1%	8.3%	21.8%	21.6%	22.1%	0.0000
		80%	40.1%	10.5%	29.6%	29.3%	30.0%	0.0000
	7	20%	10.2%	2.5%	7.6%	7.5%	7.7%	0.0000
		40%	20.1%	5.1%	15.0%	14.9%	15.2%	0.0000
		60%	30.1%	12.2%	18.0%	17.7%	18.2%	0.0000
		80%	40.4%	13.9%	26.5%	26.1%	26.8%	0.0000
8	4	20%	10.3%	1.3%	9.0%	9.0%	9.1%	0.0000
		40%	20.3%	2.6%	17.7%	17.6%	17.9%	0.0000
		60%	30.3%	5.7%	24.6%	24.3%	24.9%	0.0000
		80%	40.3%	6.3%	34.0%	33.7%	34.3%	0.0000
	7	20%	10.3%	1.3%	9.0%	9.0%	9.1%	0.0000
		40%	20.3%	4.2%	16.1%	15.9%	16.3%	0.0000
		60%	30.3%	7.6%	22.7%	22.5%	23.0%	0.0000
		80%	43.0%	8.8%	34.2%	33.8%	34.6%	0.0000
12	4	20%	10.5%	0.9%	9.6%	9.5%	9.7%	0.0000
		40%	20.5%	2.1%	18.4%	18.3%	18.6%	0.0000
		60%	30.5%	5.2%	25.2%	25.0%	25.5%	0.0000
		80%	41.0%	5.7%	35.3%	34.9%	35.6%	0.0000
	7	20%	10.5%	0.9%	9.6%	9.5%	9.7%	0.0000
		40%	20.5%	5.2%	15.3%	15.1%	15.4%	0.0000
		60%	30.8%	6.7%	24.1%	23.9%	24.4%	0.0000
		80%	45.8%	7.9%	37.9%	37.6%	38.3%	0.0000

Table B.7: Comparison on leftover inventory under UTR_3.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	2.2%	0.7%	1.5%	1.5%	1.5%	0.0000
		40%	8.4%	3.4%	5.0%	4.9%	5.1%	0.0000
		60%	18.7%	12.0%	6.7%	6.5%	6.9%	0.0000
		80%	33.2%	13.8%	19.4%	19.2%	19.6%	0.0000
	7	20%	2.3%	0.8%	1.6%	1.5%	1.6%	0.0000
		40%	9.0%	4.0%	5.0%	4.9%	5.1%	0.0000
		60%	20.1%	15.3%	4.9%	4.7%	5.1%	0.0000
		80%	35.6%	15.2%	20.4%	20.2%	20.6%	0.0000
8	4	20%	2.3%	0.4%	1.9%	1.9%	1.9%	0.0000
		40%	8.8%	2.2%	6.5%	6.4%	6.6%	0.0000
		60%	19.5%	7.4%	12.0%	11.9%	12.2%	0.0000
		80%	34.3%	9.9%	24.4%	24.2%	24.6%	0.0000
	7	20%	2.7%	0.5%	2.1%	2.1%	2.2%	0.0000
		40%	10.2%	3.9%	6.3%	6.2%	6.5%	0.0000
		60%	22.6%	9.1%	13.5%	13.3%	13.7%	0.0000
		80%	39.1%	9.6%	29.4%	29.2%	29.7%	0.0000
12	4	20%	2.6%	0.3%	2.2%	2.2%	2.3%	0.0000
		40%	9.5%	2.2%	7.3%	7.2%	7.4%	0.0000
		60%	20.8%	6.3%	14.5%	14.3%	14.7%	0.0000
		80%	36.1%	7.6%	28.5%	28.2%	28.7%	0.0000
	7	20%	3.1%	0.5%	2.6%	2.6%	2.6%	0.0000
		40%	11.4%	5.5%	5.9%	5.7%	6.0%	0.0000
		60%	25.0%	7.5%	17.5%	17.3%	17.7%	0.0000
		80%	42.0%	7.8%	34.3%	34.0%	34.5%	0.0000

Table B.8: Comparison on amount of vaccine administered under UTR_1.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	20.0%	20.0%	0.0%	0.0%	0.0%	0.0006
		40%	32.1%	37.1%	5.0%	4.9%	5.0%	0.0000
		60%	42.3%	48.5%	6.3%	6.1%	6.4%	0.0000
		80%	48.5%	48.7%	0.2%	0.1%	0.4%	0.0117
	7	20%	19.9%	19.9%	0.0%	0.0%	0.0%	0.9814
		40%	31.5%	36.6%	5.1%	5.0%	5.2%	0.0000
		60%	41.5%	44.9%	3.4%	3.2%	3.6%	0.0000
		80%	45.4%	45.7%	0.3%	0.1%	0.5%	0.0164
8	4	20%	19.9%	19.9%	0.0%	0.0%	0.0%	0.0000
		40%	31.7%	38.5%	6.7%	6.7%	6.8%	0.0000
		60%	42.0%	46.7%	4.8%	4.6%	4.9%	0.0000
		80%	46.9%	47.7%	0.9%	0.7%	1.0%	0.0000
	7	20%	19.0%	19.3%	0.3%	0.3%	0.4%	0.0000
		40%	30.6%	36.7%	6.1%	6.0%	6.3%	0.0000
		60%	39.2%	40.8%	1.6%	1.4%	1.8%	0.0000
		80%	41.6%	42.7%	1.0%	0.8%	1.3%	0.0000
12	4	20%	19.3%	19.6%	0.3%	0.2%	0.3%	0.0000
		40%	31.2%	38.5%	7.3%	7.2%	7.4%	0.0000
		60%	41.3%	44.1%	2.8%	2.6%	3.0%	0.0000
		80%	44.8%	46.2%	1.4%	1.2%	1.5%	0.0000
	7	20%	18.1%	19.4%	1.3%	1.3%	1.3%	0.0000
		40%	29.6%	34.2%	4.5%	4.4%	4.7%	0.0000
		60%	36.2%	37.3%	1.1%	0.8%	1.3%	0.0000
		80%	38.5%	39.7%	1.3%	1.0%	1.5%	0.0000

Table B.9: Comparison on amount of vaccine administered under UTR_2.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	9.8%	17.5%	7.6%	7.5%	7.7%	0.0000
		40%	19.9%	35.0%	15.1%	14.9%	15.3%	0.0000
		60%	29.9%	48.2%	18.4%	17.9%	18.8%	0.0000
		80%	39.9%	48.6%	8.7%	8.2%	9.2%	0.0000
	7	20%	9.8%	17.5%	7.6%	7.5%	7.7%	0.0000
		40%	19.9%	34.9%	15.0%	14.9%	15.2%	0.0000
		60%	29.9%	44.1%	14.2%	13.7%	14.7%	0.0000
		80%	39.6%	45.0%	5.4%	4.9%	5.9%	0.0000
8	4	20%	9.7%	18.7%	9.0%	9.0%	9.1%	0.0000
		40%	19.7%	37.4%	17.7%	17.6%	17.9%	0.0000
		60%	29.7%	46.5%	16.8%	16.4%	17.2%	0.0000
		80%	39.7%	47.4%	7.8%	7.3%	8.2%	0.0000
	7	20%	9.7%	18.7%	9.0%	9.0%	9.1%	0.0000
		40%	19.7%	35.8%	16.1%	15.9%	16.3%	0.0000
		60%	29.7%	39.8%	10.2%	9.8%	10.6%	0.0000
		80%	37.0%	42.0%	5.0%	4.5%	5.5%	0.0000
12	4	20%	9.5%	19.1%	9.6%	9.5%	9.7%	0.0000
		40%	19.5%	37.9%	18.4%	18.3%	18.6%	0.0000
		60%	29.5%	43.9%	14.4%	14.0%	14.8%	0.0000
		80%	39.0%	45.8%	6.8%	6.4%	7.3%	0.0000
	7	20%	9.5%	19.1%	9.6%	9.5%	9.7%	0.0000
		40%	19.5%	33.8%	14.3%	14.0%	14.6%	0.0000
		60%	29.2%	36.6%	7.4%	7.1%	7.8%	0.0000
		80%	34.2%	39.1%	5.0%	4.5%	5.4%	0.0000

Table B.10: Comparison on amount of vaccine administered under UTR_3.

Duration	Start	Supply	PB	PIB	Avg PB-PIB	95% CI		p-value
4	4	20%	17.8%	19.3%	1.5%	1.5%	1.5%	0.0000
		40%	31.6%	36.6%	5.0%	4.9%	5.1%	0.0000
		60%	41.3%	48.0%	6.7%	6.5%	6.9%	0.0000
		80%	46.8%	48.5%	1.6%	1.4%	1.9%	0.0000
	7	20%	17.7%	19.2%	1.6%	1.5%	1.6%	0.0000
		40%	31.0%	36.0%	5.0%	4.9%	5.1%	0.0000
		60%	39.9%	44.6%	4.7%	4.5%	5.0%	0.0000
		80%	44.4%	45.6%	1.2%	0.9%	1.5%	0.0000
8	4	20%	17.7%	19.6%	1.9%	1.9%	1.9%	0.0000
		40%	31.2%	37.8%	6.5%	6.4%	6.6%	0.0000
		60%	40.5%	46.5%	5.9%	5.7%	6.2%	0.0000
		80%	45.7%	47.4%	1.7%	1.5%	2.0%	0.0000
	7	20%	17.3%	19.5%	2.1%	2.1%	2.2%	0.0000
		40%	29.8%	36.1%	6.3%	6.2%	6.5%	0.0000
		60%	37.4%	40.5%	3.1%	2.8%	3.3%	0.0000
		80%	40.9%	42.3%	1.4%	1.1%	1.7%	0.0000
12	4	20%	17.4%	19.7%	2.2%	2.2%	2.3%	0.0000
		40%	30.5%	37.8%	7.3%	7.2%	7.4%	0.0000
		60%	39.2%	43.9%	4.7%	4.4%	5.0%	0.0000
		80%	43.9%	45.9%	2.0%	1.7%	2.3%	0.0000
	7	20%	16.9%	19.5%	2.6%	2.6%	2.6%	0.0000
		40%	28.6%	34.0%	5.3%	5.2%	5.5%	0.0000
		60%	35.0%	37.2%	2.2%	2.0%	2.5%	0.0000
		80%	38.0%	39.6%	1.7%	1.3%	2.0%	0.0000

APPENDIX C

REASSIGNMENT

C.1 Summarized Algorithms

C.1.1 Algorithm 1: upper bound

1. Calculate the total utility $TU = p\sum_{j=1}^{n} u_j$, the average remaining utility for unassigned workers $AR = \frac{TU}{n}$, and set the initialize current utility $CU = pn$. Initialize $k = 1$.

2. Assign αp times job k to worker k, and assign the remaining $(1-\alpha)p$ unassigned jobs with the least utilities to worker k. Update the following quantity:

$$\begin{aligned} TU &\leftarrow TU - CU, \\ AR &\leftarrow \frac{TU}{n-k}, \\ k &\leftarrow k+1. \end{aligned}$$

   Repeat step 2 until $CU < AR$ or $k = n$.

3. If $k = n$ then let $k^* = n$ and $ub = AR$. If $CU < AR$ then output the number of workers assigned as $k^* = k - 1$ and the upper bound as:

$$ub = \lfloor (TU + CU)/(n - k^*) \rfloor.$$

C.1.2 Algorithm 2: transform optimal solutions such that the first k^* workers are assigned the same jobs as in Algorithm 1

1. If worker k is assigned less than or equal to $i = \alpha p$ times to jobs with utility $\geq u_k$, i.e., $\sum_{s=1}^{k} M(s,k) \leq i$, go to 2. Otherwise, switch higher utility jobs assigned to worker k with lower utility jobs until k is only assigned i times to jobs with utility greater or equal to u_k.

   (a) among jobs assigned to k in M differently from Algorithm 1, identify the jobs with the smallest (a) and largest (b) index;

   (b) let w^* denote the worker with the largest index assigned to job b in M, differently from Algorithm 1;

   (c) switch job b from worker w^* to k and job a from worker k to w^*;

   (d) repeat until $\sum_{s=1}^{k} M(s,k) \leq i$.

2. Now the number of times worker k is assigned to a job with utility greater or equal to u_k is i. If worker k is assigned to job k exactly i times, go to 3. Otherwise, $M(k,k) < i \wedge \sum_{s=1}^{k} M(w_s,k) = i$. We switch out jobs with utility higher than u_k from worker k and switch in job k to worker k.

   (a) among jobs assigned to worker k in M differently from Algorithm 1, identify the jobs with the smallest (a) index;

   (b) let w^* denote the worker with the smallest index assigned with job k in M, differently from Algorithm 1;

   (c) switch job a from worker k to w^* and job k from worker w^* to worker k;

   (d) repeat until worker k is assigned to job k exactly i times.

3. Now all other jobs assigned to worker k have utility less than u_k. If the assignments are the same as in Algorithm 1, go to 4. Otherwise,

(a) identify the set of jobs J_k, to which worker k is assigned differently from Algorithm 1; identify the jobs in J_k with the smallest (j^*) and largest (j') index;

(b) identify w^*, the worker with the largest index and is assigned more periods to job j' in M than in Algorithm 1;

(c) switch job j^* out from worker k and identify the set of workers (W_k) who are involved in the pairs with stability less than α;

(d) if W_k is empty, switch job j^* to worker w^* and switch job j' from worker w^* to worker k; otherwise:

i. switch job j^* to worker w^*, the most preferred worker by job j^* in W_k;

ii. identify j'', the job with the least utility worker w^* is assigned to ($j'' > j'$ since otherwise w^* is only assigned with jobs no worse than j' and (w^*, j') would be α-stable, contradicting to $w^* \in W_k$);

iii. switch out job j'' from worker w^*, set $j' = j''$ and update W_k. Repeat (d) until W_k is empty.

(e) repeat step 3 until all assignments of k are the same as in Algorithm 1.

4. Let $k \leftarrow k + 1$ and repeat 1 to 3 until $k = k^* + 1$.

C.1.3 Algorithm 3: algorithm for $\alpha \leq \frac{1}{2}$ (static, vertical heterogeneity)

When the number of periods p is even, the assignment is easy: for each $k \in [n]$, assign worker k to job k in half of the periods and to job $n - k + 1$ in the other half. When the number of periods p is odd, the quantity $\frac{p(n+1)}{2}$ can be fractional and needed to be rounded down. We solve for when $p = 3$ and $\alpha \leq \frac{p-1}{2p}$ by assigning each worker k to jobs k, $\left\lceil \frac{n-k+1}{2} \right\rceil$, and $n - \left\lceil \frac{k}{2} \right\rceil + 1$. In the proof of Theorem 3 we show that this solution achieves the maximum of worker utility $\left\lfloor \frac{p(n+1)}{2} \right\rfloor$. If $p > 3$, we can divide the problem into two problems (problem 1 and problem 2) with the same number of workers and jobs (n), but with the number of periods $p_1 = 3$ and $p_2 = p - 3$ (even), and combine the assignments from the subproblems. This solution provides one period of stable assignment in problem 1 and $\frac{p-3}{2}$ in problem 2. Therefore, the final solution has a stability of $\frac{1+\frac{p-3}{2}}{p} = \frac{p-1}{2p}$. The proof verifies that the optimal value is achieved for both odd and even combinations of the number of agents (n) and the number of periods p.

1. If p is odd, assign worker k to job k, $n - \lceil \frac{k}{2} \rceil + 1$, and $\lceil \frac{n-k+1}{2} \rceil$. Let $p = p - 3$.

2. In the remaining periods (which is even), for each worker $k \in [n]$, assign worker k to job k and job $n - k + 1$ exactly $\frac{p}{2}$ times .

C.1.4 Algorithm 3: algorithm for $\alpha \leq \frac{1}{2}$ (static, more general preference lists)

For $i \in [n]$, obtain the utility u_{ii} of the worker-job pair (i, i) and rank workers in descending order of u_{ii}, denoted as "utility ranking" w_1 through w_n. Let j_1 be worker w_n's favorite job and for all $2 \leq k \leq n$, subsequently define j_k the favorite job of worker w_{n-k+1} excluding jobs j_1 through j_{k-1}.

Similarly we obtain two copies of job 1 through job n. Let j_1^h be worker w_n's favorite job and subsequently define j_k^h the favorite job of worker w_{n-k+1} excluding jobs j_1^h through j_{k-1}^h. Since we have two copies of each job, it is possible that $j_1^h = j_2^h$. Now there are n jobs left. For the remaining jobs, we let j_1^l be worker w_n's favorite job and j_k^l be the job worker w_{n-k+1} prefers the most excluding all jobs $j_k^h, k \in n$ and jobs j_1^l through j_{k-1}^l.

1. For each worker $i \in [n]$, obtain the index k such that $i = w_k, k \in [n]$ (if workers have identical preference lists, $i = k$), and

2. if p is odd, assign job i, j_{n-k+1}^h, and j_{n-k+1}^l exactly once to worker k and let $p \leftarrow p-3$ and go to step 3.

3. Otherwise, assign job i and job j_{n-k+1} each $\frac{p}{2}$ times.

C.1.5 Heuristic 1: heuristic for $\frac{1}{2} < \alpha \leq \frac{3}{4}$ (static, vertical heterogeneity)

1. For workers $i \leq \lceil \frac{n}{2} \rceil$, assign worker i exactly αp times to job i and $(1-\alpha)p$ times to job $n-i+1$.

2. If $\alpha = \frac{p+1}{2p}$, then p is odd and $p \geq 3$. For workers $\lceil \frac{n}{2} \rceil + 1$ to n, then

   (a) If $p > 3$, assign worker i to job i and job $n - i + \lceil \frac{n}{2} \rceil + 1$ exactly $(1-\alpha)p - 1$ times.

   (b) There are $p - 2 \cdot ((1-\alpha)p - 1) = (2\alpha - 1)p + 2 = 3$ periods left to assign. If i is odd, assign worker i to job $i, n - \left\lceil \frac{k - \lceil \frac{n}{2} \rceil}{2} \right\rceil + 1$ and $\lceil \frac{n-k+1}{2} \rceil$ exactly once. If i is even, assign worker i to job $i, n - \lceil \frac{n}{2} \rceil - \left\lceil \frac{k - \lceil \frac{n}{2} \rceil}{2} \right\rceil + 1$ and $\lceil \frac{n-k+1}{2} \rceil + \lceil \frac{n}{2} \rceil$ exactly once.

   (c) If $\alpha > \frac{p+1}{2p}$, assign worker i exactly $(1-\alpha)p$ times to job i and $n - i + 1$. Now there are $p - 2(1-\alpha)p = (2\alpha - 1)p > 1$ copy of jobs $\lceil \frac{n}{2} \rceil + 1$ through n left for workers from $\lceil \frac{n}{2} \rceil + 1$ through n. Let $n' = n - \lceil \frac{n}{2} \rceil, p' = (2\alpha - 1)p$ and subtract $\lceil \frac{n}{2} \rceil$ from the index of every worker and job. Assign the new problem with $n', p', \alpha \leq \frac{1}{2}$ and with the same preference lists using Algorithm 3. After obtaining the assignment, add $\lceil \frac{n}{2} \rceil$ to the index of every worker and job and combine with the other assignments in the first $2(1-\alpha)p$ periods.

C.1.6 Heuristic 1: heuristic for $\frac{1}{2} < \alpha \leq \frac{3}{4}$ (static, more general preference lists)

For worker $i \leq \lceil \frac{n}{2} \rceil$, we obtain the utility rankings w_1 through $w_{\lceil \frac{n}{2} \rceil}$ for workers 1 through $\lceil \frac{n}{2} \rceil$ similarly to that in section C.1.4 and let w_k be the corresponding utility ranking of worker i. Let j_1^l denote the favorite job of worker w_n in jobs $\lceil \frac{n}{2} \rceil + 1$ through n and subsequently define j_k^l the favorite job of worker w_{n-k+1} excluding jobs w_1 through w_{k-1}.

For worker $i > \lceil \frac{n}{2} \rceil$, obtain the utility ranking $w_{\lceil \frac{n}{2} \rceil + 1}$ through w_n. If n is odd (even), Let j_1^h denote the favorite job of worker w_n in jobs 1 through $\lceil \frac{n}{2} \rceil - 1$ $\left(\lceil \frac{n}{2} \rceil\right)$ and subsequently define j_k^h with odd (even) k as the favorite job of worker w_{n-k+1} in jobs 1 through $\lceil \frac{n}{2} \rceil - 1$ $\left(\lceil \frac{n}{2} \rceil\right)$, excluding jobs j_1^h through j_{k-1}^h. Define j_k^l similarly for worker w_{n-k+1} with even (odd) k and for jobs $\lceil \frac{n}{2} \rceil + 1$ through n.

1. For $i \in [n]$, let w_k be the corresponding ranked worker index for worker i. For workers $i \leq \lceil \frac{n}{2} \rceil$, assign worker i exactly αp times to job i and $(1-\alpha)p$ times to job j_{n-k+1}^l.

2. If $\alpha = \frac{p+1}{2p}$ then p is odd and $p \geq 3$. For workers $\lceil \frac{n}{2} \rceil + 1$ to n, then

   (a) If $p > 3$, assign worker i to job i and job j_{n-k+1}^h exactly $(1-\alpha)p - 1$ times.

   (b) There are $p - 2 \cdot ((1-\alpha)p - 1) = (2\alpha - 1)p + 2 = 3$ periods left to assign. Assign worker i to job i, j_{n-k+1}^h and j_{n-k+1}^l once.

3. If $\alpha > \frac{p+1}{2p}$, assign worker i exactly $(1-\alpha)p$ times to job i and j_{n-k+1}^h. Now there are $p - 2(1-\alpha)p = (2\alpha-1)p > 1$ copy of jobs $\lceil \frac{n}{2} \rceil + 1$ through n left for workers from $\lceil \frac{n}{2} \rceil + 1$ through n. Let $n' = n - \lceil \frac{n}{2} \rceil$, $p' = (2\alpha - 1)p$ and subtract $\lceil \frac{n}{2} \rceil$ from the index of every worker and job. Assign the new problem with $n', p', \alpha \leq \frac{1}{2}$ and with the same preference lists using Algorithm 3. After obtaining the assignment, add $\lceil \frac{n}{2} \rceil$ to the index of every worker and job and combine with the other assignments in the first $2(1-\alpha)p$ periods.

C.1.7 Heuristic 2: heuristic for $\alpha > \frac{3}{4}$ (static, vertical heterogeneity)

The first two steps of the heuristics try to determine the number of workers $n - x$ (which is $\geq \lceil \frac{n}{2} \rceil$) assigned similarly in Heuristic 1. This quantity is determined by α and another quantity t, where t is obtained by calculating how many "good" jobs is needed to bring the utility of worker n close to the upper bound. We refer workers 1 through $n - x$ "the higher ranking workers" and refer workers $n - x + 1$ through n "the lower ranking workers". We simultaneously assign t best jobs and $p - t$ worst jobs starting from worker n. The quantity

x is obtained by calculating where the best jobs meet the worst jobs: $\frac{tx}{(1-\alpha)p} + x = n \rightarrow x = \left\lfloor \frac{np(1-\alpha)}{t+(1-\alpha)p} \right\rfloor$.

1. Obtain an upper bound ub from Algorithm 1. Let M be an $n \times n$ assignment matrix. Temporarily assign worker n exactly p times to job n. We switch the best jobs to worker n one by one to improve the utility.

2. Let CU be the utility of worker n in the current iteration, and NU be the utility of worker n in the next iteration. Initialize $CU = p, NU = p + n$, and $t = 0$. Perform step 3, update CU, NU until $|NU - ub| > |CU - ub|$. Let $x = \left\lfloor \frac{np(1-\alpha)}{t+(1-\alpha)p} \right\rfloor$.

3. For $i \leq n - x$, assign worker i to job i exactly αp times. For $i \geq n - x + 1$, assign worker i to job i exactly $p - t$ times. Starting from worker n, assign the best t available jobs to the current worker, until all workers $i > n - x$ have been assigned. Finally for workers 1 through $n - x$, starting with worker $n - x$, assign the best available jobs one by one to the current worker, until all workers $i \leq n - x$ have been assigned.

C.1.8 Heuristic 2: heuristic for $\alpha > \frac{3}{4}$ (static, more general preference lists)

1. Obtain ub from Algorithm 1. Let M be an $n \times n$ assignment matrix. Suppose worker w is assigned to job j in the in the worker-optimal stable matching and worker w has the minimum utility. Temporarily assign worker w exactly p times to job j and let j^{least} be the least preferred job on worker w's preference list. Let j^{most} be the job with maximum utility among jobs not assigned to worker w. Switch j^{least} to j^{most} one by one to improve the utility of worker w.

2. Let CU be the utility of worker n in the current iteration, and NU be the utility of worker w in the next iteration. Initialize $CU = p, NU = p + n$, and $t = 0$. Perform step 3, update CU, NU until $|NU - ub| > |CU - ub|$.

3. For $i \leq n - x$, assign worker i to job i exactly αp times. For $i \geq n - x + 1$, assign worker i to job i exactly $p - t$ times. For workers $n - x + 1$ to n, obtain w_{n-x+1} to w_n similar in section C.1.4 with the remaining jobs. Starting from worker w_n, assign the best t available jobs to the current worker until all workers $i \leq n - x$ have been assigned. Finally for workers 1 through $n - x$, obtain w_1 to w_{n-x} similarly using the remaining jobs. Starting with worker w_{n-x}, assign the best available jobs one by one to the current worker w_k in descending order of k.

C.1.9 Heuristic 3: heuristic for $a > \frac{1}{2}$ (static, vertical heterogeneity)

1. Obtain upper bound ub and assignments of the first k^* workers from Algorithm 1.

2. For each worker $k^* + 1 \leq k \leq n$, assign jobs to meet the required stability with minimum utility and switch lower utility jobs with higher utility jobs one by one to increase the individual utility of worker k. Without loss of generality, we assume $k^* \leq n - 2$ (otherwise Algorithm 3 will give full assignment). Let $cp = 0$ be the number of periods reserved for maintaining the stable-levels.

   (a) Assign $i' = i - cp$ jobs with the least utility among jobs with utility higher or equal to k to worker k. Update the available job list and assign the $p - i$ available jobs with the least utility to worker k.

      i. If all the available jobs have utility greater or equal to k, then all pairs involving workers from k to n are α-stable. We assign jobs in descending order of utility to workers in ascending order of individual utility one by one until all workers and jobs are assigned.

      ii. Otherwise, we identify which job to switch out from worker k:

         A. among all jobs assigned to worker k, choose the one with the least utility if there is at most one job with utility more than k assigned to worker k;

B. otherwise, choose the one with the second least utility.

Obtain the current utility ut_k of worker k. Let k^{least} be the job identified above to switch out, and uw_k be the utility of job k^{least}.

(b) $ub - ut_k$ is our targeted improvement on worker k's utility. Among the available jobs, if the job with the most utility, k^{most}, has utility ub_k and $ub_k - uw_k \leq ub - ut_k$, switch k^{least} with k^{most} for worker k. Update the available job list, re-identify k^{most}, k^{least} and repeat 2.(b) until $ub_k - uw_k > ub - ut_k$. This step is incorporated in procedure BOOST.

(c) For all available jobs $a_1, ..., a_q$, obtain their utilities $u'_1, ..., u'_q$. Let k^{least} be the job with the least utility assigned to worker k, switch $k^{least}$with the following targeted job:

$$\arg\min_j \{|(u'_j - uw_k) - (ub - ut_k)|\}.$$

If there is a tie, switch to the job with maximum utility.

(d) If k is assigned to job k less than the number of times that worker $k-1$ is assigned to job $k-1$, then there is risk that workers $k+1, ..., n$ will be unstable. We design procedure PREVENT to assign worker $s, s > k$ exactly one best available jobs one by one. If there are not enough jobs better than job k available, then rollback one iteration of switching on worker k and move to the next worker. Increment the number of times PREVENT is called $cp \leftarrow cp + 1$, let $k \leftarrow k + 1$ and repeat Step 2.

C.1.10 Heuristic 4: heuristic side to middle (dynamic $p = 3, \alpha \leq \frac{1}{3}$)

1. In period one, assign worker i to job i for all $i \in [n]$ and update the workers' preference lists.

2. In period two, obtain the worker ranking w_k based on individual utility in period one

for each worker $i \in [n]$. Define j_k^h and j_k^l similarly in section C.1.4 for the updated preference lists. If i is odd, assign worker i to j_{n-k+1}^l; if i is even, assign worker i to j_{n-k+1}^h. Update workers' preference lists.

3. In period three, for workers $i \in [n]$, update the worker rankings w_k based on the total individual utility in period one and two and the index mapping $i = w_k$. Define j_1^3 the favorite job of worker w_n (with least individual utility in period one and two) and subsequently define j_k^3 as the favorite remaining job of worker w_k excluding j_1^3 through j_{k-1}^3. Assign worker i to job $j_k^3, \forall i \in [n]$.

C.1.11 Heuristic 5: heuristic side to middle (dynamic $p = 3, \frac{1}{3} < \alpha \leq \frac{2}{3}$)

1. In period one, assign worker i to job i for all $i \in [n]$ and update the workers' preference lists.

2. In period two, obtain the utility ranking w_k based on individual utility in period one for each worker $i \in [n]$. If $i \leq \lceil \frac{n}{2} \rceil$ is odd, assign worker i to job $i+1$; if $i \leq \lceil \frac{n}{2} \rceil$ is even, assign worker i to job $i-1$. Similar to that in section C.1.4, define j_k^h and j_k^l for workers $i > \lceil \frac{n}{2} \rceil$ and jobs not already assigned to workers 1 through $\lceil \frac{n}{2} \rceil$ based on the updated workers' preference lists. For $i > \lceil \frac{n}{2} \rceil$, assign worker i to job j_{n-k+1}^l and update workers' preference lists.

3. In period three, for worker $i > \lceil \frac{n}{2} \rceil$, update the worker rankings w_k based on the total individual utility in period one and two. Define j_k^3 with one copy of jobs 1 through $\lceil \frac{n}{2} \rceil$ similar to that in Heuristic 4 step 3. For worker $i > \lceil \frac{n}{2} \rceil$, define j_k^3 similarly with the remaining jobs. Assign worker $i \in [n]$ to job j_k^3.

C.2 Proof of Theorems and Corollaries

C.2.1 Theorem 1: upper bound

Proof. We prove this theorem by induction. For iteration $k = 1$, the minimum number of times job 1 is assigned to worker 1 for worker-job pair $(1, 1)$ to meet the stability requirement is αp. By assumptions on the preference lists, worker-job pair $(1, t), t \geq 1$ and $(t, 1), t \geq 1$ have stability α. For $k \geq 2$, assume that every worker $t < k$ is assigned to job t exactly αp times and the job with the least utility is $> k$. Then there are several ways to satisfy the stability requirement for pair (k, k):

1. worker k is assigned to job k exactly αp times,
2. worker k is assigned to job 1 through k exactly αp times, and
3. worker k is assigned to job 1 through k exactly $n_k < \alpha p$ times, job k prefers some workers from 1 through $k-1$ to worker k and job k is assigned to these jobs exactly $\alpha p - n_k$ times.

Note that Case 2 always assigns more or the same utility to workers from 1 to k compared to Case 1, so Case 2 is undesirable. In Case 3, we can assume that worker k is assigned to job k exactly n_k times. Worker k is assigned to jobs with the least utility $p - n_k$ times and some workers from 1 through $k-1$ that job k prefers than worker k are assigned less than $(1-\alpha)p$ times to jobs with the least utility. If we calculate the total utility of jobs assigned to workers from 1 to k, it is the same compared to Case 1, which is what Algorithm 1 does.

Furthermore, for two consecutive workers s and $s+1$, the average remaining utility is decreasing:

$$\begin{aligned} AR_s - AR_{s+1} &= \frac{AR_{s+1}(n-s-1) + CU_{s+1}}{n-s} - AR_{s+1} \\ &= \frac{AR_{s+1}(n-s) + CU_{s+1} - AR_{s+1}}{n-s} - AR_{s+1} \\ &= \frac{CU_{s+1} - AR_{s+1}}{n-s} > 0, \end{aligned}$$

which gives $CU_s \geq AR_s > AR_{s+1}$. In other words, every worker is assigned to utility greater than the final output AR_{k^*}. Therefore, Algorithm 1 gives an upper bound on the objective function value. □

C.2.2 Theorem 2: correctness of transform algorithm

Proof. Suppose a given assignment matrix M is optimal and at least one of the first k^* workers are not assigned the same as in Algorithm 1. For $k = 1$, the number of jobs assigned to worker k with utility greater or equal to u_k is at least αp since M is α-stable. We inductively show that in every step of Algorithm 2, the number of switches is finite, and the resulting new matching is still α-stable. Since at the end of the algorithm, workers $k \leq k^*$ are assigned the same as in Algorithm 1, their individual utilities are greater or equal to the upper bound. We only need to show that for workers $> k^*$, the utility is still greater or equal to the upper bound.

1. For step 1:

   - w^* is assigned with a better job (a over b);
   - the stability of all pairs involving job a are met by the inductive hypothesis on stable assignment: $M(s,s) = \alpha p, s < k$ and $\sum_{s=1}^{k} M(s,k) = \alpha p$;
   - the stability of all pairs involving job b are met since job b has the least utility among all jobs assigned to worker $k, ..., n$.

   For each k, there are at most $(1-\alpha)p$ jobs we need to switch in this step.

2. For step 2:

   - w^* is assigned with better jobs (a over k);
   - the stability of all pairs involving job a and k are met since $a < k$ and $M(s,s) = \alpha p, s \in [k]$.

   For each k, there are at most αp jobs we need to switch in this step.

3. For step 3:

   - in each iteration, w^* is assigned to a better job;
   - in each iteration, assigning j' to other workers does not affect the stability of pairs involving job j' by definition of W_k.

   For each k, we can switch at most $(1-\alpha)p$ jobs and each job can be switched at most $n - k^*$ times.

The number of switches performed in this algorithm is at most $k^*((1-\alpha)p + \alpha p + (1-\alpha)p(n-k^*)) \leq n(p+pn) = np(n+1)$. Therefore the complexity of this algorithm is $O(n^2 p)$. □

C.2.3 Theorem 11: $\mathcal{NP}$-hardness reduction

Theorem 11. *Consider the following decision problem:*

> *When workers have the same preference lists, jobs have arbitrary preference lists, and the utility of job k, $u_k \in \mathbb{Z}^+ \cup \{0\}, \forall k \in [n]$ satisfies $u_1 \geq \cdots \geq u_n \geq 0$, is there an α-stable p-period assignment to the multi-period stable matching problem such that the minimum utility over all workers is greater or equal to $K \in \mathbb{Z}$?*

This decision problem is $\mathcal{NP}$-complete.

The proof uses the fact that when the number of periods p is sufficiently large and $\alpha = \frac{p-1}{p}$, the number of workers assigned by Algorithm 1, k^*, can be exactly $n-2$, and worker k is assigned to job k exactly $p-1$ times for $k \in [n-2]$. The remaining two workers, $n-1$ and n, are left with each of jobs 1 through n - 2 and several jobs $n-1$ and n. If the utilities of jobs $n-1$ and n are both 0, this specific problem is equivalent to the partition problem (with numbers equal to the utilities of job 1 through $n-2$).

Proof. We reduce the $\mathcal{NP}$-complete partition problem to the multi-period stable matching problem with a particular stability requirement:

> Given a set of distinct positive integers $S = \{a_i, i \in [m]\}$ with descending order of $a_k \in \mathbb{Z}^+, \forall k \in [m]$, is there a partition of S such that the sum of the integers in each subset equal to K?

Let $n = m+2$, $u_k = a_k, \forall k \in [m]$, $u_{m+1} = u_{m+2} = 0$, $p = \max\left\{\left\lceil \frac{\sum_{k=1}^{m} a_m}{2a_m} \right\rceil + 1, m\right\}$, $\alpha = \frac{p-1}{p}$, and subset sum $K = \left\lfloor \frac{\sum_{k=1}^{m} a_k}{2} \right\rfloor$. Running Algorithm 1 and checking the stopping condition, we have $k^* = m$.

1. If the partition problem has a "yes" answer, let $S = S_1 \cup S_2$ be the solution. For workers $k \in [m]$, assign $p-1$ times job k to worker k and job $m+2$ to worker k one time. Assign jobs indicated in S_1 and $p - s_1$ times job $m+1$ to worker $m+1$. Assign jobs indicated in S_2 and the remaining jobs to worker $m+2$. It's easy to see that the assignment is $\frac{p-1}{p}$-stable, since every worker k is assigned to job k at least $p-1$ times for $k \in [m]$, and every job that worker $m+1$ and $m+2$ are assigned to are no worse than job $m+1$ and $m+2$, respectively. The utility of worker $m+1$ and $m+2$ is K, and the utilities of workers 1 through m are larger than K.

2. If the multi-period stable matching problem has a "yes" answer with an assignment matrix M, by Theorem 2, M can be transformed into another optimal solution M' with the property that $M'(k,k) = p-1, \forall k \in [m]$. Since the minimum utility over

all workers is K, the assignments of workers $m+1$ and $m+2$, excluding jobs $m+1$ and $m+2$ form a partition of the multi-set S, with the sum of each subset being K.

Therefore, a partition problem can be reduced to a multi-period stable matching problem.

□

In the proof, we assumed that the last job is strictly less preferred by all workers compared to other jobs. However since the last two jobs have utility 0, it could be more reasonable to assume that all workers are indifferent between the last two jobs (ties). In weak stable matchings, a worker-job pair not in matching M is a blocking pair if each member of the pair prefers the other to their partner in matching M. A matching without such blocking pairs is weakly stable. In several papers, it was shown that a matching M is weakly stable in a preference list P with ties, if it is stable in at least one instance P' of stable matching which can be derived from P by randomly breaking the ties [107, 108, 109]. We expect our results to hold under ties if we define α-stable matchings in multiple periods similarly.

C.2.4 Theorem 3: correctness of Algorithm 3 (static, reverse rank utility, $\alpha \leq \frac{1}{2}$)

Proof. When p is even and workers have identical preference lists, the total utility of jobs is

$$\frac{p}{2}(u_i + u_{n-i+1}) = \frac{p(n+1)}{2}, \forall i \in [n],$$

which equals to the optimal value since every worker has the same utility. The stability for each worker-job pair (i,i) is at least $\frac{p}{2}$ since $M(i,i) \geq \frac{p}{2}, \forall k \in [n]$. Recall that the utility ranking of worker i is w_k and in the unique stable assignment, worker i is assigned to a job with utility at least $n-k+1$. Similarly, the utility of j_k^h and j_k^l is at least $n - \left\lceil \frac{n-k+1}{2} \right\rceil + 1$ and $\left\lceil \frac{k}{2} \right\rceil$.

When p is odd, let us first consider $p = 3$: the utility received by worker i is at least $n-k+1$ (for job i), $n - \left(\left\lceil \frac{n-k+1}{2} \right\rceil\right) + 1$ (for j_{n-k+1}^h) and $\left\lceil \frac{k}{2} \right\rceil$ (for j_{n-k+1}^l). The total utility for worker i is at least $2n-k+2-\left\lceil \frac{n-k+1}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil$.

1. when n is even, the optimal value is $\left\lfloor \frac{3(n+1)n}{2n} \right\rfloor = \frac{3n+2}{2}$.

   (a) k is even, the total utility of worker i is at least

   $$2n-k+2-\frac{n-k+2}{2}+\frac{k}{2}=\frac{3n+2}{2};$$

   (b) k is odd, then the total utility of worker i is at least

   $$2n-k+2-\frac{n-k+1}{2}+\frac{k+1}{2}=\frac{3n+4}{2};$$

2. n is odd, the optimal value is $\left\lfloor \frac{3(n+1)n}{2n} \right\rfloor = \frac{3n+3}{2}$.

   (a) k is even, the total utility of worker i is at least

   $$2n-k+2-\frac{n-k+1}{2}+\frac{k}{2}=\frac{3n+3}{2};$$

   (b) k is odd, the total utility of worker i is at least

   $$2n-k+2-\frac{n-k+2}{2}+\frac{k+1}{2}=\frac{3n+3}{2};$$

When workers have identical preference lists, the utility of worker i is greater or equal to the optimal value in every case. For odd $p \geq 5$, Algorithm 3 first gives a three-period assignment, then adds a $p-3$ (even) period assignment. Both assignments give equal utility (maximum difference on utility between any worker is 1) to all workers. For preference lists satisfying the SPC conditions, each worker i has utility at least the same as that of worker i when workers have identical preference lists.

Note that switching the assignments of worker k with worker $n-k+1$ for all $k \leq \left\lfloor \frac{n}{2} \right\rfloor$ also gives an optimal solution by symmetry of the assignments. $\square$

C.2.5 Theorem 4: correctness of Heuristic 1 (static, $\frac{1}{2} < \alpha \leq \frac{3}{4}$) on more general preference lists

Proof. From Heuristic 1, $M(k,k) = \alpha p, \forall k \leq \lceil \frac{n}{2} \rceil$. By condition (4.9) and (4.10), all worker-job pairs $(k,s), s \leq \lceil \frac{n}{2} \rceil \vee k \leq \lceil \frac{n}{2} \rceil$ have stability α, as workers are assigned to their stable assignments. The only pairs left to evaluate are $(k,s), k,s > \lceil \frac{n}{2} \rceil$.

1. If $\alpha = \frac{p+1}{2p}$, each worker $i, i > \lceil \frac{n}{2} \rceil$, is assigned $(1-\alpha)p$ times to jobs from 1 through $\lceil \frac{n}{2} \rceil$, with utility at least $n - \lceil \frac{n}{2} \rceil + 1$ (since $k > n-k+1$), and $(1-\alpha)$ times to job i. The stability for any remaining pair is at least $2(1-\alpha) = \frac{p-1}{p} \geq \alpha$.

2. Otherwise, $\alpha > \frac{p+1}{2p}$, and each worker $k, k \geq \lceil \frac{n}{2} \rceil$ is assigned $(1-\alpha)p$ times to jobs from 1 through $\lceil \frac{n}{2} \rceil$ and $(1-\alpha) + \left\lfloor \frac{p-2(1-\alpha)p}{2} \right\rfloor = \lfloor \frac{p}{2} \rfloor$ times to job i. Therefore, any pair has stability at least $\left((1-\alpha)p + \lfloor \frac{p}{2} \rfloor\right)/p$. To satisfy the stability α for all pairs, we need

$$\frac{(1-\alpha)p + \lfloor \frac{p}{2} \rfloor}{p} \geq \alpha \tag{C.1}$$

$$\implies 1 - \alpha + \left\lfloor \frac{p}{2} \right\rfloor /p \geq \alpha \tag{C.2}$$

$$\implies \frac{1 + \lfloor \frac{p}{2} \rfloor /p}{2} \geq \alpha \tag{C.3}$$

If p is even, then $\alpha \leq \frac{3}{4}$ is obviously satisfied. If p is odd, then we need $\alpha \leq \frac{3}{4} - \frac{1}{4p}$. We know that when p is odd, taking $\alpha = \frac{3}{4}$ is the same as taking $\alpha = \frac{q}{p}$, where

$$\frac{q}{p} \leq \frac{3}{4} \leq \frac{q+1}{p} \tag{C.4}$$

$$\implies 4q \leq 3p \leq 4q + 4. \tag{C.5}$$

Since p is odd, the quantity $3p$ is odd. Also since $4q$ and $4q+4$ are both even, we can further tighten the inequality as

$$4q + 1 \leq 3p \leq 4q + 3 \tag{C.6}$$

$$\implies \frac{q}{p} + \frac{1}{4p} \leq \frac{3}{4} \leq \frac{q}{p} + \frac{3p}{4}. \tag{C.7}$$

Therefore when p is odd, requiring $\alpha \leq \frac{3}{4}$ is the same as requiring $\alpha \leq \frac{3}{4} - \frac{1}{4p}$.

□

C.2.6 Theorem 12: correctness of Heuristic 2 (static, $\alpha > \frac{3}{4}$) on more general preference lists

Theorem 12 (Correctness of Heuristic 2)**.** *For $p \geq 3$, Heuristic 2 gives an α-stable assignment for $\alpha > \frac{3}{4}$.*

Proof. Heuristic 2 assigns αp times job i to worker i for $i \in [n-x]$, so worker-job pairs (s, q) with $s \in n-x$ or $q \in n-x$ are α-stable by conditions (4.9) and (4.10). For any other pair (s, q) with $s < q, s > n-x$ and $q > n-x$, worker s is either assigned to job s or a job $r \leq n-x$, both better than job q. For any pair $(s, q), s > q, s > n-x$ and $q > n-x$, worker s is assigned to job $r \leq n-x$ (better than job q) t times and job q is assigned to worker q (better than worker s) $p-t$ times. Therefore, the stability for (s, q) is 1 for $s > n-x, q > n-x$ and all worker-jobs pairs are α-stable. □

C.2.7 Theorem 13: correctness of Heuristic 3 on vertical heterogeneity

Theorem 13 (Correctness of Heuristic 2 on vertical heterogeneity)**.** *For $p \geq 3$, Heuristic 3 gives an α-stable assignment for $\alpha > \frac{1}{2}$.*

Proof. For pairs involving workers $k \leq k^*$, the stability requirements are satisfied because worker k is assigned to job k in αp periods. For the rest of pairs involving workers $k > k^*$, the stability requirements are satisfied because worker k is assigned to job k in i' periods, where $i' = i - cp$, and worker k is assigned to jobs better than job k in cp periods. □

C.2.8 Theorem 5: upper bound calculation (static, $\alpha > \frac{1}{2}$) on vertical heterogeneity

Proof. Let $q = \alpha p$. By construction of Algorithm 1, when the algorithm terminates we have the following two conditions.

1. The total utility worker k^* receives over p periods is at least the average remaining utility at step k^* (denoted as ub):

$$\begin{aligned}(n-k^*)(q(n-k^*+1)+\lceil k^*(p-q)/p\rceil(p-q)) \geq \\ (pn(n+1)/2 - qk^*(2n-k^*+1)/2- \\ (p-q)(1+\lfloor k^*(p-q)/p\rfloor)(\lfloor k^*(p-q)/p\rfloor/2)- \\ (\lceil k^*(p-q)/p\rceil)p(k^*(p-q)-\lfloor k^*(p-q)/p\rfloor)).\end{aligned} \tag{C.8}$$

2. The total utility worker k^*+1 receives over p periods is at most the average remaining utility at step k^*+1 (if carried out):

$$\begin{aligned}(n-(k^*+1))(q(n-k^*)+\lceil (k^*+1)(p-q)/p\rceil(p-q)) \leq \\ (pn(n+1)/2 - q(k^*+1)(2n-k^*)/2- \\ (p-q)(1+\lfloor (k^*+1)(p-q)/p\rfloor)(\lfloor (k^*+1)(p-q)/p\rfloor/2)- \\ (\lceil (k^*+1)(p-q)/p\rceil)p((k^*+1)(p-q)-\lfloor (k^*+1)(p-q)/p\rfloor)).\end{aligned} \tag{C.9}$$

Let $\beta = \lim_{n\to\infty} \frac{k^*}{n}$ be the percentage of workers assigned in Algorithm 1, then from (C.8) to (C.9) we have:

$$\begin{aligned}n^2(1-\beta)(q(1-\beta)+\beta(p-q)^2/p) = \\ pn(n+1)/2 - n^2q\beta(2-\beta)/2 - n^2\beta^2(p-q)^2/2p,\end{aligned} \tag{C.10}$$

and β can be obtained by solving the following equation:

$$(p^2 - 3pq + q^2)\beta^2 - 2(p^2 - 3pq + q^2)\beta + (p^2 - 2pq) = 0, \tag{C.11}$$

which gives

$$\beta = 1 - \sqrt{\frac{q(p-q)}{pq-(p-q)^2}} = 1 - \sqrt{\frac{\alpha(1-\alpha)}{3\alpha - \alpha^2 - 1}}. \tag{C.12}$$

Let $UB(\alpha) = \lim_{n\to\infty} \frac{ub}{np}$, then

$$UB(\alpha) = \frac{1}{p}\left(q(1-\beta) + \frac{\beta(p-q)^2}{p}\right) = (1-\alpha)^2 + \sqrt{\alpha(1-\alpha)(3\alpha-\alpha^2-1)}. \tag{C.13}$$

□

C.2.9 Corollary 6: analytical performance of Heuristic 1 (static, $\frac{1}{2} < \alpha \leq \frac{3}{4}$) on vertical heterogeneity

Proof. For workers $i \leq \lceil \frac{n}{2} \rceil$, let $w_k = i$ be the corresponding utility ranking. The utility of worker w_k is at least:

$$\alpha p(n-k+1) + (1-\alpha)pk = p(\alpha n + \alpha - (2\alpha - 1)k), \tag{C.14}$$

which is a decreasing function on k since $\alpha > \frac{1}{2}$, and the minimum utility is at least the utility of worker $w_{\lceil \frac{n}{2} \rceil}$:

$$u_{min1} = p\left(\alpha n + \alpha - (2\alpha - 1)\left\lceil \frac{n}{2} \right\rceil\right). \tag{C.15}$$

For workers $i > \lceil \frac{n}{2} \rceil$, every worker has a utility at least $\lfloor u_{min2} \rfloor$, which is:

$$u_{min2} = \frac{(1-\alpha)p\left(n + \cdots + \lceil \frac{n}{2} \rceil\right) + \alpha p \left(\lceil \frac{n}{2} \rceil - 1 + \cdots + 1\right)}{\lceil \frac{n}{2} \rceil} \tag{C.16}$$

$$= \frac{(1-\alpha)p\left(n + \lceil \frac{n}{2} \rceil\right)\lceil \frac{n}{2} \rceil + \alpha p \lceil \frac{n}{2} \rceil \left(\lceil \frac{n}{2} \rceil - 1\right)}{2\lceil \frac{n}{2} \rceil} \tag{C.17}$$

$$= \frac{1}{2}p\left((1-\alpha)\left(n + \left\lceil \frac{n}{2} \right\rceil\right) + \alpha\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\right) \tag{C.18}$$

$$= \frac{1}{2}p\left(n + \left\lceil \frac{n}{2} \right\rceil - \alpha n - \alpha\right). \tag{C.19}$$

We now compare u_{min1} and u_{min2}:

$$u_{min1} - u_{min2} = p\left(\alpha n + \alpha - (2\alpha - 1)\left\lceil \frac{n}{2} \right\rceil\right) - \frac{1}{2}p\left(n + \left\lceil \frac{n}{2} \right\rceil - \alpha n - \alpha\right) \tag{C.20}$$

$$= p\left(\alpha n + \alpha - (2\alpha - 1)\left\lceil \frac{n}{2} \right\rceil - \frac{1}{2}\left(n + \left\lceil \frac{n}{2} \right\rceil - \alpha n - \alpha\right)\right) \tag{C.21}$$

$$= \frac{1}{2}p\left(3\alpha n + 3\alpha - (4\alpha - 1)\left\lceil \frac{n}{2} \right\rceil - n\right) \tag{C.22}$$

$$\geq \frac{1}{2}p\left(3\alpha n + 3\alpha - 4\alpha\frac{n+1}{2} + \frac{n-1}{2} - n\right) \tag{C.23}$$

$$= \frac{1}{2}p\left(3\alpha n + 3\alpha - 2\alpha n - 2\alpha + \frac{n}{2} - \frac{1}{2} - n\right) \tag{C.24}$$

$$= \frac{1}{2}p\left(\alpha n + \alpha - \frac{1}{2}n - \frac{1}{2}\right) = \frac{1}{2}p\left(\alpha - \frac{1}{2}\right)(n+1) > 0 \tag{C.25}$$

Therefore we have u_{min2} as the minimum individual utility. For large enough n, the assignment returned by Heuristic 1 gives a minimum utility (normalized by n, p) of

$$u_{he}(\alpha) = \lim_{n \to \infty} u_{min2} = \frac{3}{4} - \frac{1}{2}\alpha. \tag{C.26}$$

We then have that the optimality gap between $u_{he}(\alpha)$ and the normalized (by n, p) optimal value, $OPT(\alpha)$, is

$$G(\alpha) = \frac{OPT(\alpha) - u_{he}(\alpha)}{OPT(\alpha)} \tag{C.27}$$
$$= 1 - \frac{u_{he}(\alpha)}{OPT(\alpha)} \tag{C.28}$$
$$\leq 1 - \frac{u_{he}(\alpha)}{UB(\alpha)} \tag{C.29}$$
$$= 1 - \frac{\frac{3}{4} - \frac{1}{2}\alpha}{\sqrt{\alpha(1-\alpha)(3\alpha - \alpha^2 - 1))} + (1-\alpha)^2} \tag{C.30}$$

The less than or equal sign follows from:

$$UB(\alpha) \geq OPT(\alpha) \to \frac{1}{UB(\alpha)} \leq \frac{1}{OPT(\alpha)} \to -\frac{1}{UB(\alpha)} \geq -\frac{1}{OPT(\alpha)}. \tag{C.31}$$

The maximum of $G(\alpha)$ is achieved at $\alpha \approx 0.74$, with a value of 11.07%. □

C.2.10 Lemma 1: for proof of Theorem 7

Lemma 1 (Linearity of worker utility)**.** *In Heuristic 2, let uw_n be the utility obtained for worker w_n, uw_x be the utility of worker w_{n-x}, and let uw_l be the utility of worker $w_l, l = n - \left\lfloor \frac{x}{2} \right\rfloor$. If workers have identical preference lists, then*

$$\lim_{n\to\infty} \frac{uw_l}{np} \geq \min\left\{ \lim_{n\to\infty} \frac{uw_n}{np}, \lim_{n\to\infty} \frac{uw_x}{np} \right\}. \tag{C.32}$$

Proof. By construction of Heuristic 2 and the property of floor and ceiling functions, we have

$$(n-p)t + p - t \leq uw_n \leq nt + p - t \tag{C.33}$$

$$tx + (p-t)(x-1) \leq uw_x \leq px \tag{C.34}$$

$$\tag{C.35}$$

Similarly for worker l, we have

$$\left(\frac{n-x-1}{2}\right)t+\left(\frac{x-1}{2}\right)(p-t)\leq uw_l\leq\left(\frac{n-x+1}{2}\right)t+\left(\frac{x+1}{2}\right)(p-t) \tag{C.36}$$

Dividing everything by np and taking $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{uw_l}{np}=\frac{1}{2}\left(\lim_{n\to\infty}\frac{uw_n}{np}+\lim_{n\to\infty}\frac{uw_x}{np}\right). \tag{C.37}$$

Therefore inequality (C.32) holds, which means the minimum utility is attained either by worker $n-x$ or worker n and every worker has a higher utility when preference lists are SPC. $\square$

C.2.11 Theorem 7: performance of Heuristic 2 (static, $\alpha > \frac{3}{4}$) on vertical heterogeneity

Proof. By Lemma 1, either worker n or worker $n-x$ attains the minimum utility when workers have identical preference lists. This is also the minimum utility of workers when the preference lists satisfy SPC. In Heuristic 2, the total utility of worker n is within $\frac{n}{2}$ of the upper bound by definition of t:

$$ub-\frac{n}{2}\leq uw_n\leq ub+\frac{n}{2}. \tag{C.38}$$

Therefore the normalized utility of worker n is:

$$UB(\alpha)-\frac{1}{2p}\leq\lim_{n\to\infty}\frac{uw_n}{np}\leq UB(\alpha)+\frac{1}{2p} \tag{C.39}$$

Also by definition of t,

$$ub-\frac{n}{2}\leq(n-p)t\leq uw_n\leq nt\leq ub+\frac{n}{2}. \tag{C.40}$$

Dividing both sides by np and taking $n \to \infty$ we get:

$$UB(\alpha) - \frac{1}{2p} \leq \frac{t}{p} \leq UB(\alpha) + \frac{1}{2p} \tag{C.41}$$

By definition of x, the total utility of worker $n - x$ satisfies:

$$p\left(\frac{n(p-i)}{t+p-i} - 1\right) \leq px \leq uw_x \leq p(x+1) \leq p\left(\frac{n(p-i)}{t+p-i} + 1\right). \tag{C.42}$$

Dividing both sides by np and taking $n \to \infty$ we get:

$$\lim_{n\to\infty} \frac{uw_x}{np} \geq \frac{1-\alpha}{UB(\alpha) + \frac{1}{2p} + 1 - \alpha} \tag{C.43}$$

□

C.2.12 Lemma 2: dynamic problem is polynomially solvable for $p \leq 2$

Lemma 2. *The dynamic problem can be solved efficiently when $p \leq 2$.*

Proof. We define job j_1^2 as the favorite job of worker n and subsequently j_k^2 as worker $n-k+1$'s favorite job, excluding jobs j_1^2 through j_{k-1}^2. In period two we assign job j_k^2 to worker k for all $k \in [n]$. It is easy to see that job j_k^2 provides at least $n-k+1$ utility to worker k, and the objective value for this assignment is $n-k+1+k = n+1$ and optimal. □

C.2.13 Lemma 3: correctness of Heuristic 4 (dynamic, $p = 3, \alpha \leq \frac{1}{3}$)

Lemma 3. *For $p = 3$ and $\alpha \leq \frac{1}{3}$ Heuristic 4 returns an α-stable solution.*

Proof. When $\alpha \leq \frac{1}{3}$, worker k is assigned to job k for every $k \leq \lceil \frac{n}{2} \rceil$, so every worker-job pair in period one is $\frac{1}{3}$-stable and Heuristic 4 returns a $\frac{1}{3}$-stable matching. □

C.2.14 Theorem 14: correctness of Heuristic 5 (dynamic, $p = 3$, $\frac{1}{3} < \alpha \leq \frac{2}{3}$)

Theorem 14. *For $p = 3$ and $\alpha \geq \frac{1}{3}$ Heuristic 5 returns an α-stable solution.*

Proof. When $\frac{1}{3} < \alpha \leq \frac{2}{3}$ and n is even (odd).

1. In period one, worker i is assigned to job $i, \forall i \in [n]$, which is the unique stable assignment.

2. In period two, worker $i \leq \left\lceil \frac{n}{2} \right\rceil$ is assigned to the favorite job excluding all jobs assigned to workers $k < i$. At the same time, these jobs are assigned to their favorite workers excluding all workers assigned to jobs with lower indexes and the worker assigned in period one. Therefore, all worker-job pairs (i, j) with $i \leq \left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)$ or $j \leq \left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)$ are stable.

3. In period three, worker $i > \left\lceil \frac{n}{2} \right\rceil$ are assigned to jobs from 1 to $\left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)$ since they are better jobs compared to jobs from $\left\lceil \frac{n}{2} \right\rceil + 1 \left(\left\lceil \frac{n}{2} \right\rceil\right)$ to n in workers' preference lists. Therefore worker-job pairs (i, j) with $i \geq \left\lceil \frac{n}{2} \right\rceil + 1$ and $j \geq \left\lceil \frac{n}{2} \right\rceil + 1$ are α-stable. As a result, workers from 1 to $\left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)$ are assigned to jobs from $\left\lceil \frac{n}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)$ to n.

4. Now the only pairs left for evaluating stability are $\left(\left\lceil \frac{n}{2} \right\rceil, k\right), k \geq \left\lceil \frac{n}{2} \right\rceil$ and $\left(k, \left\lceil \frac{n}{2} \right\rceil\right), k \geq \left\lceil \frac{n}{2} \right\rceil$ when n is odd. Worker $\left\lceil \frac{n}{2} \right\rceil$ is assigned to job $\left\lceil \frac{n}{2} \right\rceil$ in period one and $\left\lceil \frac{n}{2} \right\rceil + 1$ in period two by construction of the heuristic. In period one, no pair is a blocking pair. In period two, pairs $\left(\left\lceil \frac{n}{2} \right\rceil, k\right), k \geq \left\lceil \frac{n}{2} \right\rceil + 1$ cannot be blocking pairs by condition 4.10 and pairs $\left(k, \left\lceil \frac{n}{2} \right\rceil\right), k \geq \left\lceil \frac{n}{2} \right\rceil + 1$ cannot be blocking pairs by condition 4.9. Pair $\left(\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right)$ is not a blocking pair because worker $\left\lceil \frac{n}{2} \right\rceil$ was assigned to job $\left\lceil \frac{n}{2} \right\rceil$ in period one, which is the least favorite job in period two.

$\square$

C.2.15 Theorem 8: upper bound for the dynamic problem

Proof. In period one, job j provides a utility of $n - j + 1$ to any worker. In period two, if job j is assigned to worker i and worker i was assigned to job $q < j$ in period one, then job j provides utility $\min\{n - j + 2, n\}$; if worker i was assigned to job j, then job j provides utility 1; if worker i was assigned to job $q > j$, then job j provides the same utility $n-j+1$ as in period one. It is easy to see that the maximum utility job j provides in period two is $\min\{n - j + 2, n\}$, which is at most 1 more than what job j could provide in period one. Making the same argument inductively on the number of periods p, we have the maximum potential utility U_{max} of job j can provide to any worker in period t is

$$U_{max}(j, t) = \min\{n, n - j + t\}. \tag{C.44}$$

We then calculate an upper bound UB_{dy} for the dynamic problem by taking the simple average of the maximum total potential utility. This upper bound is independent of the stability α, with n workers, n jobs and p periods.

$$UB_{dy}(n, p) = \left\lfloor \frac{1}{n} \sum_{j=1}^{n} \sum_{t=1}^{p} U_{max}(j, t) \right\rfloor \tag{C.45}$$

First we analyze whether or when the upper bound is tight. To attain the upper bound UB_{dy}, we need to attain U_{max} for each job j in each period p. In general, assigning job j to a worker i who has been assigned to better jobs $s < j$ in previous periods increase the potential utility of job j in the current period, because jobs s was moved to the bottom of the preference list of worker i. In period one, each job j attains $U_{max}(j, 1) = n - j + 1$. In period two, equation (4.16) shows that in order to achieve the maximum potential utility, exactly $n-1$ jobs need to improve their utility by 1. With job 1 already attains the maximum utility, we need every job from 2 to n to be assigned to workers who was assigned a better job in period one. The only feasible assignment that meets this requirement is to assign worker j to job $j+1$ for $j \leq n-1$ and assign job 1 to worker n. By induction on period t, there exists an unique assignment that attains $U_{max}(j,t)$ for any job j and period t, which we summarize below as the "rotating assignment".

1. In period one, assign worker k to job k and let $N(1,k) = k, \forall k \in [n]$.

2. In period $t \geq 2$, we have the assignment of each worker k in the previous period $N(t-1,k)$. We assign worker k to job $N(t-1,k)+1$ if $N(t-1,k)+1 \leq n$ and to job 1 otherwise.

We can view the rotating assignment as rotating jobs over workers with higher ranked orders in each period. At the same time, the assignment shuffles the preference lists of workers. In period $t \geq 2$, job $j > t$ is assigned to worker $j-t+1$, which decreases as t increases. From period one through t, job $j > t$ attains the least utility among jobs assigned to worker $j-t+1$. Therefore, job $j > t$ attains $U_{max}(j,t) = n-j+t$. For job $j < t$, the maximum utility m is attained as job j is the favorite job in period t for the worker job j is assigned to. The rotating assignment is stable in period one but not in other periods. After $t = n$ periods, the preference lists of workers satisfy Horizontal heterogeneity and all subsequent assignments are stable. Therefore for a p period problem, the rotating assignment only provides a $\frac{1}{p}$-stable solution when $p < n$. $\square$

C.3 Formulation of the dynamic problem, $p = 3, \alpha = \frac{2}{3}$

Decision variables:

For $p = 3, i, j \in [n], k \in [p]$, and $\alpha = \frac{2}{3}$

$x_{ij}^k = 1$, if worker i is assigned to job j in period i, binary variable.

$c_{ij}^k =$ the amount of utility from job j for worker i in period k, non-negative continuous.

$s_{ij}^k =$ the stability generated from pair (i,j) in period k, non-negative continuous.

Objective function: maximizing U.

Constraints:

1. Assignment constraints: $\forall i \in [n], k \in [p]$

$$\sum_{i=1}^{n} x_{ij}^{k} = 1 \tag{C.46}$$

$$\sum_{j=1}^{n} x_{ij}^{k} = 1 \tag{C.47}$$

2. Utility constraints: $\forall i, j \in [n]$ and $k \in [p]$

$$c_{ij}^{k} \leq (n - j + k) \cdot x_{ij}^{k} \tag{C.48}$$

$$\text{If } x_{ij}^{1} = 1 \text{ then } c_{ij}^{2} \leq 1 \tag{C.49}$$

$$\text{If } x_{ij}^{2} = 1 \text{ then } c_{ij}^{3} \leq 1 \tag{C.50}$$

$$\text{If } x_{ij}^{1} = 1 \text{ then } c_{ij}^{3} \leq 2 \tag{C.51}$$

$$\text{If } \sum_{p<j} x_{ij}^{1} = 0 \text{ then } c_{ij}^{2} \leq (n - j + 1) \cdot x_{jk}^{2} \tag{C.52}$$

$$\text{If } \sum_{p<j} x_{ij}^{1} + x_{ij}^{2} = 0 \text{ then } c_{ij}^{3} \leq (n - j + 1) \cdot x_{jk}^{3} \tag{C.53}$$

$$\text{If } \sum_{p=1}^{j-1} x_{ij}^{1} + x_{ij}^{2} = 1 \text{ then } c_{ij}^{3} \leq (n - j + 2) \cdot x_{jk}^{3} \tag{C.54}$$

$$\text{If } \sum_{p<j} x_{ij}^{1} + x_{ij}^{2} = 2 \text{ and } \exists j \text{ s.t. } x_{ij}^{1} + x_{ij}^{2} = 2 \text{ then } c_{ij}^{3} \leq (n - j + 2) \cdot x_{jk}^{3} \tag{C.55}$$

$$\text{If } \sum_{p=1}^{j-1} x_{ij}^{1} + x_{ij}^{2} = 2 \text{ and } \forall j, x_{ij}^{1} + x_{ij}^{2} \leq 1 \text{ then } c_{ij}^{3} \leq (n - j + 3) \cdot x_{jk}^{3} \tag{C.56}$$

3. Stable constraints: for all $i, j \in [n]$

$$s_{ij}^{1} \leq \sum_{p \leq i} x_{pk}^{1} + \sum_{q<j} x_{iq}^{1} \tag{C.57}$$

$$\sum_{k=1}^{p} s_{ij}^{k} \geq 3\alpha, \forall i, j \in [n] \tag{C.58}$$

$$\text{If } x_{ij}^1 = 1 \text{ then } s_{ij}^2 \leq 1 + \sum_{p \leq i} x_{pj}^2 \tag{C.59}$$

$$\text{If } x_{il}^1 = 1 \text{ and } l < j \text{ then } s_{ij}^2 \leq \sum_{p \leq i} x_{pj}^2 + \sum_{2 \leq q < j} x_{iq}^2 - x_{il}^2 \tag{C.60}$$

$$\text{If } x_{il}^1 = 1 \text{ and } l > j \text{ then } s_{ij}^2 \leq \sum_{p \leq i} x_{pj}^2 + \sum_{2 \leq q < j} x_{iq}^2 \tag{C.61}$$

$$\text{If } x_{ij}^2 = 1 \text{ then } s_{ij}^3 \leq \sum_{p \leq i} x_{pj}^3 \tag{C.62}$$

$$\text{If } x_{ij}^1 = 1 \text{ and } x_{ij}^2 = l \text{ then } s_{ij}^3 \leq \sum_{q \neq l, q \neq j} x_{iq}^3 + \sum_{p \leq i} x_{pj}^3 \tag{C.63}$$

$$\text{If } x_{il}^1 = 1, x_{im}^2 = 1, l < j, m < j \text{ then } s_{ij}^3 \leq \sum_{p \leq i} x_{pj}^2 + \sum_{q < j} x_{iq}^2 - x_{il}^2 - x_{im}^2 \tag{C.64}$$

$$\text{If } x_{il}^1 = 1, x_{im}^2 = 1, l < j \text{ and } m > j \text{ then } s_{ij}^3 \leq \sum_{p \leq i} x_{pj}^2 + \sum_{q < j} x_{iq}^2 - x_{il}^2 \tag{C.65}$$

$$\text{If } x_{il}^1 = 1, x_{im}^2 = 1, l > j \text{ and } m < j \text{ then } s_{ij}^3 \leq \sum_{p \leq i} x_{pj}^2 + \sum_{q < j} x_{iq}^2 - x_{im}^2 \tag{C.66}$$

$$\text{If } x_{il}^1 = 1, x_{im}^2 = 1, l > j, \text{ and } m > j \text{ then } s_{ij}^3 \leq \sum_{p \leq i} x_{pj}^2 + \sum_{q < j} x_{iq}^2 \tag{C.67}$$

4. Maximizing minimum utility

$$\sum_{j=1}^{n} \sum_{k=1}^{p} c_{ij}^k \geq U \tag{C.68}$$

C.4 Job categories

The dynamic preference lists discussed in section 4.6 considers the situation in which worker moves the previous assignments to the bottom of the preference lists. When the number of job is large, the preference change may not be significant for small periods. In this section, we consider the situation where nm jobs can be divided into n categories with the same number of jobs m in each category. The dynamics on workers' preference changes as follows: if workers are assigned to a job in category c in the current period, then in the next period all jobs in category c will move to the bottom of workers' preference lists. In this case, we allow the preference lists of workers change more significantly in one period. We refer this new problem "the job category problem". The relative ranked orders of jobs inside the same category do not change over time. We assume workers have identical preference lists in period one and the utility function is reverse rank utility. The dynamic preference lists discussed in section 4.6 is a special case with one job in each job category.

The most significant difference between the dynamic problem and the job category problem is the magnitude of change in workers' preference lists. In a way, we can view good solutions from the dynamic problem as efficient mechanics to rotate job categories for all workers. Depending on the number of jobs in each job category (m), just using the same job rotation as the smaller dynamic problem could be sufficient in obtaining a good assignment for the job category problem. To quantify the performance of this simple derived heuristic, we first analyze the relations of the optimal solution between the two problems.

Suppose we have a feasible assignment N, with p rows and nm columns and each entry (i, j) indicating the job assigned to worker j in period i. Consider another matrix N' with the same size and each entry indicating the job category assigned to each worker in each period. For each column k of the assignment matrix N, we define two utility scores:

1. Job utility $s_k \in [p, pnm]$, the total utility of jobs for worker k in all periods, and

2. Category utility $t_k \in [p, pn]$, the total utility of job categories for worker k in all periods.

For any worker $k \in [n]$ with category utility t_k, we calculate the range of job utility s_k. In the best scenario, worker k is assigned to the best job in the corresponding job categories that give a category utility t_k. The maximum job utility is then $mp \cdot t_k$. Similar in the worst scenarios, worker k is assigned to the worst job in each period. Since the number of job in each job category is m, the difference between the maximum and minimum job utility for the same category utility s_k is $(m-1)p$. Therefore, the minimum job utility is $mp \cdot t_k - (m-1)p = mp \cdot (t_k - 1) + p$. The range of job utility s_k for the same category utility t_k is

$$mp \cdot (t_k - 1) + p \leq s_k \leq mp \cdot t_k. \tag{C.69}$$

With the quantified relation between s_k and t_k, we show that workers with a lower job utility always has a lower category utility.

Theorem 15. *For all* $1 \leq k_1, k_2 \leq nm$*, if* $s_{k_1} \leq s_{k_2}$*, then* $t_{k_1} \leq t_{k_2}$

Proof. Suppose there exists $1 \leq k_1, k_2 \leq nm$ such that $s_{k_1} \leq s_{k_2}$ and $t_{k_1} \geq t_{k_2} + 1$. By inequality (C.69) we have:

$$s_{k_2} \geq s_{k_1} \geq (t_{k_1} - 1)mp + p \geq t_{k_2}mp + p > t_{k_2}mp \geq s_{k_2} \tag{C.70}$$

which is a contradiction. Therefore, $s_{k_1} \leq s_{k_2} \implies t_{k_1} \leq t_{k_2}$. □

Since $s_{k_1} \leq s_{k_2} \implies t_{k_1} \leq t_{k_2}$ is equivalent to $t_{k_1} > t_{k_2} \implies s_{k_1} > s_{k_2}$, we can relate the optimal solutions of the two problems using the following Corollary 16.

Corollary 16. *If we have an optimal solution with value* t^* *for the dynamic problem with size* n*, then there exists an optimal solution of the job category problem with all workers* $s \in [nm]$ *having* $t_s \geq t^*$*.*

Proof. Immediate from Theorem 15. □

Corollary 16 clears the way for designing Heuristic C.4 that utilizes the job category rotations by solving a size n dynamic problem and then assigns workers to jobs within the same job category according to the worker-optimal stable matching.

1. Let a p by n matrix N be an α-stable assignment for the size n dynamic problem. Expand each column of n by m times, with columns $(l-1)m+1$ through lm having the same values as the original column l.

2. Create a p by nm matrix N'. For each worker j obtain the remainder $r_j = j \mod m$ and assign $N'(k,j) = (N(k,j)-1)\cdot m + m - r_j = N(k,j)\cdot m - r_j$.

Across the job categories, the stability requirement is satisfied since N is α-stable. Within job category, we assign the worker-optimal stable matchings and all pairs are α-stable. The solution has the property that workers $(l-1)m+1$ through lm have the same category utility for all $l \in [n]$. When the number of jobs in each job category (m) is large, the preference lists for workers can be significantly different. As a result, it is harder to come up with sufficient conditions similar to that in Corollary 10.

Next we quantify the performance of the job category heuristic. Let $UB_{dy}(n,p)$ be the upper bound value for the dynamic problem with size n and period p. Let S be the objective function value returned from the dynamic heuristics. For m workers with category utility t, the max-min utility is the average of the maximum and minimum value:

$$Opt = \frac{(mp(t-1)+p+mpt)m}{2m} = mpt - \frac{(m-1)p}{2} \tag{C.71}$$

By inequality (C.69) we have that

$$Gap = 1 - \frac{(S-1)mp+p}{mpUB - \frac{(m-1)p}{2}} = 1 - \frac{S-1+\frac{1}{m}}{UB - \frac{m-1}{2m}} \tag{C.72}$$

Notice that if $m=1$ then the equation reduces to $Gap = 1 - \frac{S}{UB}$, which is exactly the gap between the dynamic heuristics and the dynamic upper bound. As m goes to infinite, we have

$$\lim_{m\to\infty} Gap = 1 - \frac{S-1}{UB - \frac{1}{2}}. \tag{C.73}$$

This quantity is very close to the original gap $1 - \frac{S}{UB}$ as when m goes to infinity, n has to go to infinite and S and UB are very large compared to 1 and $\frac{1}{2}$.

C.5 Pseudocode of algorithms, heuristics, and procedures

In this section we provide example pseudocode for the algorithms and heuristics developed in Chapter 4. We only present the pseudocode for when workers and jobs have identical preference lists. Below we list commonly used notations in the pseudocodes.

1. "$++(--)$": adding (subtracting) one to (from) the current variable
2. "$+= x(-= x)$": adding (subtracting) x to (from) the current variable
3. $M(i,:)$: the i^{th} row of matrix M; $M(:,j)$ is the j^{th} column of matrix M

Algorithm 1 Procedure to find upper bound on the objective function value when the workers have the same preference lists

procedure INITIALIZATION(n, i, p)
    $M \leftarrow n \times n$ matrix with each entry 0;
    $TU \leftarrow p \sum_{j=1}^{n} u_j$;
    $AR \leftarrow TU/n$;
    $k \leftarrow 1$;
    $CU \leftarrow pn$;
    while $\lfloor AR \rfloor \leq CU$ **do**
        $M(k,k) \leftarrow i$;
        $index \leftarrow \lfloor (p-i)(k-1)/p \rfloor$;
        $Remain \leftarrow (p-i)(k-1) - p \cdot index$;
        if $Remain \leq i$ **then**
            $M(k, n-index) \leftarrow p-i$;
            $CU \leftarrow i(n-k+1) + u_{n-index}(p-i)$;
        else
            $M(k, n-index) \leftarrow p - Remain$;
            $M(k, n-index-1) \leftarrow Remain - i$;
            $CU \leftarrow i(n-k+1) + u_{n-index}(p-Remain) + u_{n-index-1}(Remain-i)$;
        end if
        $TU \leftarrow TU - CU$;
        $AR \leftarrow TU/(n-k)$;
        $k++$;
    end while
    $k^* \leftarrow k-1$;
    $ub \leftarrow \lfloor (TU+CU)/(n-k^*) \rfloor$;
end procedure

Algorithm 2 Transform any optimal solution to one with the assignments of the first k^* workers specified in Algorithm 1. This algorithm uses procedure SWITCH (see Appendix).

Input: k^*, M

for $1 \leq k \leq k^*$ **do**

while $\sum_{s=1}^{k} M(s,k) > i$ **do**

$a \leftarrow \arg\min_s \{s : s > k, M(k,s) \neq M_0(k,s)\}$;

$b \leftarrow \arg\max_s \{s : s > k, M(k,s) \neq M_0(k,s)\}$;

$w^* \leftarrow \arg\max_w \{w : w \neq k, M(w,b) > M_0(w,b)\}$;

$M \leftarrow \text{SWITCH}(M(k), M(w^*), a, b)$;

end while

while $M(k,k) < i \wedge \sum_{s=1}^{k} M(w_s,k) = i$ **do**

$a \leftarrow \arg\min_s \{s : s > k, M(k,s) \neq M_0(k,s)\}$;

$w^* \leftarrow \arg\min_w \{w : w \neq k, M(w,k) > M_0(w,k)\}$;

$M \leftarrow \text{SWITCH}(M(k), M(w^*), a, k)$;

end while

$J_k \leftarrow \{s : s > k, M(k,s) \neq M_0(k,s)\}$;

while $J_k \neq \emptyset$ **do**

$j^* \leftarrow \arg\min_s \{s : s \in J_k\}$;

$j' \leftarrow \arg\max_s \{s : s \in J_k\}$;

$w^* \leftarrow \arg\max_w \{w : w \neq k, M(w,j') > M_0(w,j')\}$;

$M(k,j^*) \leftarrow M(k,j^*) - 1$;

$W_k \leftarrow \{w : \sum_{j=1}^{k} M(w,j) + \sum_{s \neq w, s >_{j^*} w^*} M(w,j^*) < i\}$;

while $W_k \neq \emptyset$ **do**

$w^* \leftarrow \{w : w \in W_k, w >_{j^*} s, \forall s \neq w\}$;

$M(w^*,j^*) \leftarrow M(w^*,j^*) + 1$;

$j' \leftarrow \arg\max_s \{s : M(w^*,s) \geq 1, u_s < u_{j^*}\}$;

$M(w^*,j') \leftarrow M(w^*,j') - 1$;

$W_k \leftarrow \{w : \sum_{j=1}^{j'} M(w,j) + \sum_{s \neq w, s >_{j'} w} M(w,j') < i\}$;

$j^* \leftarrow j'$;

end while

$M(k,j') \leftarrow M(k,j') + 1$;

$J_k \leftarrow \{s : s > k, M(k,s) \neq M_0(k,s)\}$;

end while

end for

$M' \leftarrow M$;

Output: M';

Algorithm 3 Algorithm to find equal utility among workers when $\alpha \leq \frac{\lceil p/2 \rceil}{p}$

procedure EQUALUTILITY(n, p)
    $M \leftarrow n \times n$ matrix with each entry zero;
    if $p \pmod 2 = 1$ **then**
        for $1 \leq k \leq n$ **do**
            $M(k, k) \leftarrow 1;$
            $M\left(k, \left\lceil \frac{n-k+1}{2} \right\rceil\right) \leftarrow 1;$
            $M\left(k, n - \left\lceil \frac{k}{2} \right\rceil + 1\right) \leftarrow 1;$
        end for
        $p \leftarrow p - 3;$
    end if
    for $1 \leq k \leq n$ **do**
        $M(k, k) \leftarrow M(k, k) + \frac{p}{2};$
        $M(k, n - k + 1) \leftarrow M(k, n - k + 1) + \frac{p}{2};$
    end for
end procedure

Heuristic 1 Heuristic for utility is reverse rank and $\alpha \leq \frac{3}{4}$

Input: $n, i, p, \frac{\lceil p/2 \rceil}{p} \leq \alpha \leq \frac{3}{4}$;
for $1 \leq k \leq \lceil \frac{n}{2} \rceil$ **do**
    $M(k,k) \leftarrow i$;
    $M(k, n-k+1) \leftarrow p - i$;
end for
if $2i - p = 1$ **then**
    if $p > 3$ **then**
        $M(k, n-k+1) \leftarrow p - i$;
        $M(k,k) \leftarrow p - i - 1$;
    end if
    for $\lceil \frac{n}{2} \rceil + 1 \leq k \leq n$ **do**
        if $n \mod 2 = 0$ **then**
            $M\left(k, \left\lceil \frac{n-k+1}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil\right) + +$;
            $M\left(k, n - \left\lceil \frac{k - \lceil \frac{n}{2} \rceil}{2} \right\rceil + 1\right) + +$;
        else
            $M\left(k, \left\lceil \frac{n-k+1}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil\right) + +$;
            $M\left(k, n - \left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{k - \lceil \frac{n}{2} \rceil}{2} \right\rceil + 1\right) + +$;
        end if
    end for
else
    $M' \leftarrow \textsc{EqualUtility}(\left\lfloor \frac{n}{2} \right\rfloor, 2i - p)$;
    for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n$; **do**
        $M(k, n-k+1) \leftarrow p - i$;
        $M(k,k) \leftarrow p - i$;
        for $1 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$ **do**
            $M\left(k, s + \left\lceil \frac{n}{2} \right\rceil\right) \leftarrow M\left(k, s + \left\lceil \frac{n}{2} \right\rceil\right) + M'\left(k - \left\lceil \frac{n}{2} \right\rceil, s\right)$;
        end for
    end for
end if

Heuristic 2 Heuristic for utility is reverse rank $\alpha > \frac{3}{4}$

Input: $n, i, p, \alpha > \frac{3}{4}$;

$ub \leftarrow \textsc{Initialization}(n, i, p)$;
$M \leftarrow n \times n$ zero matrix, except entry (n, n) is p;
$free(1) \leftarrow p - i - 1$;
$free(k) \leftarrow p - i, \forall k \geq 2$;
$CU \leftarrow p$; ▷ utility of worker in the current iteration
$NU \leftarrow p + n$; ▷ utility of worker in the next iteration
$t \leftarrow 0$;
while $|NU - ub| \leq |CU - ub|$ **do**
    $index \leftarrow \arg\min_k \{k : free(k) > 0\}$;
    $free(index) - -$;
    $CU \leftarrow NU$;
    $NU + = n - index + 1$; ▷ update current and next utility
    $t + +$;
end while
$x \leftarrow \left\lfloor \frac{n(p-i)}{t+p-i} \right\rfloor$; ▷ number of workers treated as lower ranked
$free(k) \leftarrow p, \forall k \in [p]$;
for $1 \leq s \leq n - x$ **do**
    $M(s, s) + = i$;
    $free(s) - = i$;
end for
for $s = n - x + 1$ to n **do**
    $M(s, s) + = p - t$;
    $free(s) - = p + t$;
    for $1 \leq k \leq t$ **do**
        $index \leftarrow \arg\min_k \{k : free(k) > 0\}$;
        $M(s, index) + +$;
        $free(index) - -$;
    end for
end for
for $s = n - x$ to 1 **do**
    for $1 \leq k \leq p - i$ **do**
        $index \leftarrow \arg\min_k \{k : free(k) > 0\}$;
        $M(s, index) + +$;
        $free(index) - -$;
    end for
end for

Heuristic 3 Heuristic for utility is reverse rank $\alpha > \frac{1}{2}$

Input: $n, i, p, \alpha = \frac{i}{p} > \frac{1}{2}$

$k^*, ub, M \leftarrow$ INITIALIZATION$(n, i, p), prevent \leftarrow 0$;
$free(1 : k^*) \leftarrow n - \sum_{s=1}^{n} M(s, 1 : k^*)$;
for $k^* + 1 \leq j \leq n - 1$ **do**
    $i' \leftarrow i - prevent$; ▷ number of jobs needed to satisfy stability
    $temp \leftarrow$ zero vector with length n;
    for $1 \leq s \leq i'$ **do** ▷ satisfy stability with minimum utility
        $index \leftarrow \arg\max_q \{q : free(q) > 0, q \leq k\}$;
        $temp(index) + +$;
        $free(index) - -$;
    end for
    for $1 \leq k \leq p - i$ **do** ▷ assign the rest jobs with minimum utility
        $index \leftarrow \arg\max_q \{q : free(q) > 0\}$;
        $temp(index) + +$;
        $free(index) - -$;
    end for
    $j' \leftarrow \arg\min_q \{q : free(q) > 0\}$; ▷ identify which jobs to switch
    $j^* \leftarrow \arg\max_q \{q : temp(q) > 0\}$;
    $CU \leftarrow \sum_{q=1}^{n} u_q(temp(q) + M(k, q))$;
    if $j^* \leq k$ **then**
        $M' \leftarrow M(k : n, :)$;
        $free, M' \leftarrow$ MIN2MAX$(free, temp, n, p, M')$;
        $M(k : n, :) + = M'$;
        break
    else
        if $|s : free(s) > 0, s > k| \leq 1$; **then**
            $index \leftarrow j^*$; ▷ switch out job with least utility
        else
            $index \leftarrow \arg\max_q \{q : temp(q) > 0, q \neq j^*\}$;
        end if
        $temp, free \leftarrow$ BOOST$(index, j', ub, CU, temp, free)$;
        $set \leftarrow \arg\min_q \{q : temp(q) > 0 \cap \arg\min_q(|index - q| - (ub - CU))\}$;
        $temp, free \leftarrow$SWITCH$(temp, free, index, set)$;
        $ub \leftarrow \min \left\{ ub, \left\lfloor n(n+1)/((2p - \sum_{s=1}^{k-1} \sum_{q=1}^{n} M(s, q))(n - k + 1)) \right\rfloor \right\}$;
        if $temp(k) < i - prevent$ **then**
            $free, prevent, M, temp \leftarrow$ PREVENT$(n, i, p, k, free, M, temp)$;
        end if
    end if
    $M(k) \leftarrow M(k) + temp$; ▷ Update assignment of worker k.
end for

Output: M;

Heuristic 4 Heuristic for dynamic preference list $p = 3, \alpha \leq \frac{1}{3}$

Input: n, P (n by n matrix with each row workers' preference list)

$N \leftarrow 3 \times n$ zero matrix;
$avail \leftarrow$ size n vector with each entry 2;
for $1 \leq k \leq n$ **do**
    $N(1, k) \leftarrow k$;
    $index \leftarrow \arg\{P(k, :) = k\}$;
    UPDATE(P);
end for
for $1 \leq c \leq 2n$ **do**
    if $c \leq n$ **then**
        $k = c$;
    else
        $k = c - n$;
    end if
    $best \leftarrow \arg\min_s\{avail(s) > 0\}$;
    $index \leftarrow \arg\{P(k, :) = best\}$;
    if $(n \mod 2 = 1 \wedge k \leq \lceil \frac{n}{2} \rceil) \vee (n \mod 2 = 0 \wedge k > \lceil \frac{n}{2} \rceil)$ **then**
        $N(2, k) \leftarrow P(k, index)$;
    else
        $N(3, k) \leftarrow P(k, index)$;
    end if
    $avail(s) - -$;
    UPDATE(P)
end for

Output: N;

Heuristic 5 Heuristic for dynamic preference list $p = 3, \alpha > \frac{1}{3}$

Input: n, P (n by n matrix with each row workers' preference list)

$N \leftarrow 3 \times n$ zero matrix;
for $1 \leq k \leq n$ **do**
  $N(1, k) \leftarrow k$;
  $index \leftarrow \arg\{P(k, :) = k\}$;
  UPDATE(P);
  if $n \mod 2 = 1 \wedge n \leq \lceil \frac{n}{2} \rceil$ **then**
    $N(2, k) \leftarrow k + 1$;
  else if $n \mod 2 = 1 \wedge n \leq \lceil \frac{n}{2} \rceil$ **then**
    $N(2, k) \leftarrow k - 1$;
  end if
end for
if $n \mod 2 = 1$ **then**
  $index \leftarrow \arg\min_k\{P(\lceil \frac{n}{2} \rceil, k) > \lceil \frac{n}{2} \rceil\}$;
  $N(3, \lceil \frac{n}{2} \rceil) \leftarrow index$;
  $avail \leftarrow$ size n vector with entries 2 except $index$.
else
  $avail \leftarrow$ size n vector with each entry 1.
end if
for $1 \leq c \leq 2(n - \lceil \frac{n}{2} \rceil)$ **do**
  if $c \leq n - \lceil \frac{n}{2} \rceil$ **then**
    $k = c + \lceil \frac{n}{2} \rceil$;
  else
    $k = c - (n - \lceil \frac{n}{2} \rceil) + \lceil \frac{n}{2} \rceil$;
  end if
  if $(k \mod 2 = 1 \wedge c \leq n - \lceil \frac{n}{2} \rceil) \vee (k \mod 2 = 0 \wedge c > n - \lceil \frac{n}{2} \rceil)$ **then**
    $best \leftarrow \arg\min_{s \leq \lceil \frac{n}{2} \rceil}\{avail(s) > 0\}$;
    $index \leftarrow \arg\{P(k, :) = best\}$;
    $N(3, k) \leftarrow P(k, index)$;
  else
    $best \leftarrow \arg\min_{s > \lceil \frac{n}{2} \rceil}\{avail(s) > 0\}$;
    $index \leftarrow \arg\{P(k, :) = best\}$;
    $N(2, k) \leftarrow P(k, index)$;
  end if
  $avail(best) - -$;
  UPDATE(P)
end for

Output: N;

Procedure 1 Switching job i from worker v_1 to v_2 and job j from worker v_2 to v_1

procedure SWITCH(v_1, v_2, i, j)
  $v_1(i) - -;$
  $v_2(i) + +;$
  $v_1(j) + +;$
  $v_2(j) - -;$
end procedure

Procedure 2 Boost: part of Heuristic 3 for improving performances

procedure BOOST($index, j', ub, CU, temp, free$)
  while do$(index - j') < (ub - CU)$
    if $j' - index \geq 0$ **then**
      break
    else
      $temp(j', 1) + +;$
      $temp(index, 1) - -;$
      $free(j') - -;$
      $free(index) + +;$
      $j' \leftarrow \arg\min_q \{q : free(q) > 0\};$
      $vec \leftarrow \{l : temp(l) = 1\};$
      $CU = \sum_{k=1}^{n} k(temp(k) + M(j, k));$
      if then$\sum_k (vec(k) > j) \leq 1;$
        $index = vec(end);$
      else
        $index = vec(length(vec) - 1);$
      end if
    end if
  end while
end procedure

Procedure 3 Prevent: part of Heuristic 3 for guaranteeing stability

procedure PREVENT($n, i, p, j, free, M, temp$)
  for $1 \leq z \leq i - prevent - temp(j)$ **do**
    if $\sum_{k=1}^{j-1} free(j) \geq n - j$ **then**
      for $j + 1 \leq k \leq n$ **do**
        $vec \leftarrow \{l : freeJobs(l) > 0\}$;
        $M(k, vec(1)) + +$;
        $free(vec(1)) - -$
      end for
      $prevent + +$;
    else
      $vec \leftarrow \{l : temp(l) > 0, l \leq j - 1\}$;
      $temp(vec(end), 1) - -$;
      $temp(j, 1) + +]$
      $free(j) - -$;
      $free(vec(end)) + +$;
    end if
  end for
end procedure

Procedure 4 Min2Max: the last part of Heuristic 3 that assign the best available jobs to workers with the minimum utility

procedure PREVENT($n, p, j, free, M'$)
  $assign \leftarrow zeros(j, 1) + \sum_{k=1}^{n}(M'(:, k))$;
  $indFree \leftarrow \{l : free(l) > 0\}$;
  for $1 \leq k \leq n$ **do**
    $rank(k) = \sum_{k=1}^{n} k \cdot M(k, :)$;
  end for
  for $1 \leq s \leq length(indFree)$ **do**
    for $1 \leq k \leq free(indFree(s))$ **do**
      $index1 \leftarrow \{l : rank(l) = \min_m(rank(assign(m) < p))\}$;
      $index2 \leftarrow \{l : assign(l) < p\}$;
      $index \leftarrow index1 \cap index2$;
      $index \leftarrow index(end)$;
      $M'(index, indexFree(s)) + +$;
      $rank(index) + = n - indFree(s) + 1$;
      $assign(index) + +$;
    end for
    $free(indFree(s)) \leftarrow 0$;
  end for
  $minRank \leftarrow min_l(\sum_k k \cdot M'(l, k))$;
end procedure

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